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Defects and bulk perturbations of boundary Landau-Ginzburg orbifolds

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Abstract: We propose defect lines as a useful tool in the study of bulk perturbations of conformal field theory, in particular in the analysis of the induced renormalisation group flows of boundary conditions. As a concrete example we study bulk perturbations of $N = 2$ supersymmetric minimal models. To these perturbations we associate a special class of defects between the respective UV and IR theories, whose fusion with boundary conditions indeed reproduces the behaviour of the latter under the corresponding RG flows.

Keywords: Renormalization Group, Topological Field Theories, D-branes.

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Contents

1. Introduction

In string theory, generic non-supersymmetric backgrounds contain tachyons and are therefore unstable. What happens to such backgrounds under the condensation of these tachyons is a very interesting question, which is however difficult to answer in general. For a special class of non-supersymmetric orbifolds this has been studied in [1−3], where use was made of the fact that the tachyons in these models are localised on space-time defects. This allows to apply techniques similar to those employed in the treatment of open string tachyon condensation.

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From the world sheet point of view, tachyon condensation is decribed by a perturbation of the conformal field theory associated to the initial background by relevant operators. The end point of the induced bulk renormalisation group flow is a new conformal field theory which describes the vacuum reached after the decay of the original background.

Unfortunately, the analysis of such bulk RG flows is very tedious and in particular for models of interest in string theory, not much is known about them. In some cases however there are additional structures which can be used in the analysis. For instance, the non-supersymmetric orbifolds mentioned above, although not being space-time supersymmetric, exhibit $N = 2$ world sheet supersymmetry, which gives more controle over the RG flow due to non-renormalisation theorems.

An interesting question which arises in the context of closed string tachyon condensation is the fate of the D-branes in the initial theory once the background decays to a new vacuum. For the non-supersymmetric orbifolds this has been addressed in [4 − 6]. Since the tachyon condensation in these examples partially resolves the orbifold singularity, there are fewer D-brane charges available after the condensation, and some of the D-branes have to decouple from the theory.

From a world sheet point of view, the effect of closed string tachyon condensation on D-branes is described by perturbations with relevant bulk fields in the presence of world sheet boundaries. Such perturbations induce flows in both the bulk as well as the boundary sectors of the theory, which makes them even more tedious to analyse.

In this paper, we propose a new approach to the regularisation and renormalisation of bulk perturbations in the presence of boundaries. As explained in some detail in section 2, we decouple bulk and boundary flows. We first perform the bulk flow and, in a second step, treat the effect on the boundary sectors, which then amounts to merging a world sheet defect line with a boundary.

Defects lines are one-dimensional interfaces which separate two possibly different conformal field theories (see e.g. [7 − 15]). A special type of such defects, so called *topological defects* can be shifted on the world sheet and in particular can be moved smoothly on top of other defects resulting in new *fused* defects. Likewise, they can be brought to world sheet boundaries transforming the original boundary conditions to different ones. Generic, non-topological defects on the other hand, cannot be moved on the world sheet without changing correlation functions, and bringing them close to world sheet boundaries (or to other such defects) results in singularities.

The defects which emerge in our treatment of bulk perturbations of boundary conformal field theories are defects between the UV and IR CFTs of the corresponding RG flows. As such they are non-topological in general, and hence their fusion with world sheet boundaries is singular. This is indeed expected. Encoding the effect of the bulk perturbation near the boundary, the process of merging the defect with the boundary has to be regularised just as the bulk perturbation near the boundary has to be.

To avoid dealing with this regularisation, we consider supersymmetric flows between $N = 2$ superconformal field theories here. The corresponding defects are supersymmetric and are in particular compatible with a topological twist of the theory. In the twisted theory, they become topological and can therefore be merged with world sheet boundaries

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without the need of regularisation, determining in this way to what boundary condition a given one flows under the bulk perturbation.

The concrete examples we study here are RG flows between orbifolds of $N = 2$ superconformal minimal models generated by twist field perturbations. These models have an alternative description as Landau-Ginzburg orbifolds which lends itself easily to the construction of supersymmetric defects and the analysis of their properties as e.g. fusion. We generalise the formalism developed in [16 − 18] to deal with B-type defects in Landau-Ginzburg models to the case of Landau-Ginzburg orbifolds, and use it to construct the defects describing the twist field perturbations of minimal model orbifolds. Indeed, these flows are very similar to the flows of non-superymmetric orbifolds mentioned above, and the methods we describe here easily generalise to perturbations of orbifolds $\mathbb{C}/\mathbb{Z}_d$.

The use of defects to describe the effect of bulk flows on boundary conditions is non-perturbative in nature. After all, it involves a defect between UV and IR conformal field theories. Thus, it is in general difficult to find the defect describing a particular perturbation. In the examples at hand however, one can use mirror symmetry to relate perturbations and defects. The mirrors of the minimal model orbifolds are the unorbifolded minimal models, which have a Landau-Ginzburg description. The twist fields perturbing the minimal model orbifolds are mapped under mirror symmetry to monomials in the chiral superfield deforming the superpotential of the Landau-Ginzburg model. The effect of such deformations on A-branes can easily be studied and compared to the fusion of defects in the Landau-Ginzburg orbifolds.

This paper is organised as follows. In section 2 we describe in some more detail how the effect of bulk perturbations on boundary conditions is captured by the fusion with a defect. In section 3 we introduce the concrete examples (perturbations of orbifolds of $N = 2$ superconformal field theories), in which we apply this method, and at the same time outline our strategy and summarise the results obtained in the following sections. Section 4 is devoted to a general discussion of B-type defects in Landau-Ginzburg orbifolds and their description in terms of equivariant matrix factorisations. In section 5 we use this formalism to construct a special class of defects between orbifolds of $N = 2$ minimal models, which we propose to be the defects arising in RG flows between these models. We also analyse their properties, in particular their fusion with each other and with B-type boundary conditions. In section 6 we compare this fusion with the behaviour of A-type D-branes under the corresponding flows on the mirror side, which can be studied rather explicitly. Section 7 contains some comments on the description of RG flows for non-supersymmetric orbifolds $\mathbb{C}/\mathbb{Z}_d$. We close with some open problems in section 8.

2. Bulk flows and defects

The topic of this article is the behaviour of D-branes or conformal boundary condition under relevant bulk perturbations. This subject has been addressed in the literature before, using the Thermodynamic Bethe Ansatz [19] the truncated conformal space method [20] or by analysing the RG flow equations for bulk and boundary couplings [21, 22]. The new idea

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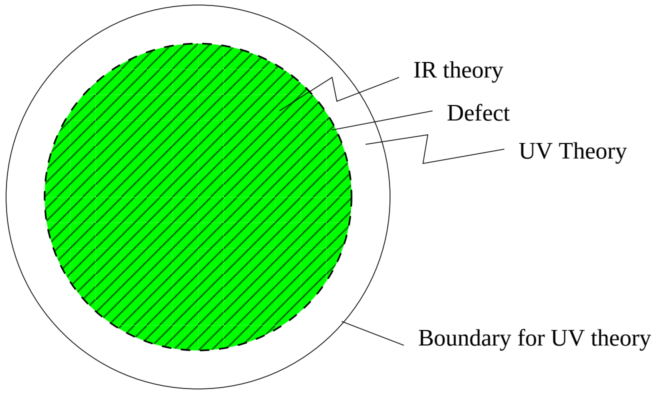

Figure 1: Perturbation restricted to a domain U (shaded). UV and IR theory are separated by a defect line.

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put forward in this article is to use defects to describe the effect of bulk perturbations of conformal field theories on conformal boundary conditions.

Conformal field theories can be perturbed by adding terms

$$\Delta S = \sum_i \lambda^i \int_\Sigma \mathrm{d}^2 z \varphi_i(z, \bar{z}) \tag{2.1}$$

to the action. Here λ^i are coupling constants, and φ_i are marginal or relevant fields which are integrated over the world sheet surface Σ. Perturbed correlation functions are then obtained from those of the unperturbed theory by

$$\langle \ldots \rangle_{\lambda^i} = \langle \ldots e^{\Delta S} \rangle_{\lambda^i = 0} \,. \tag{2.2}$$

Obviously, expressions like this have to be regularised for instance by means of a cutoff restricting the integration domain of the perturbations away from any other field insertion. The renormalisation group flow then drives the system from the UV to the IR fixed point (if existent) of the perturbation, which is another conformal field theory. In the special case where the operator is marginal, the theory remains conformal for all values of the coupling constants.

Instead of a perturbation on the entire surface, one can also consider perturbations which are restricted to a domain $U \subset \Sigma$, as indicated in figure 1. In the same way as before, one obtains perturbed correlation functions and a renormalisation group flow. Since local properties outside U are not affected by the perturbation, the correlation functions at the endpoint of the flow describe the situation of the original UV conformal field theory on $\Sigma - U$ and the IR theory on U, which because of conformal invariance are separated by a conformal defect line on the boundary ∂U. In this way, perturbations give rise to conformal defects separating UV and IR theories of the corresponding renormalisation group flows. (Similarly, perturbations with exactly marginal operators give rise to defects between the unperturbed and the perturbed theory.)

This relation between bulk RG flows and defects is particularly useful in the study of bulk perturbations of conformal field theories on surfaces with boundary.[1] Apart from the

[1] See [21, 23, 24] for recent discussions of bulk induced boundary flows using other methods.

bulk RG flow such perturbations in general also induce RG flows in the boundary sectors, and UV boundary conditions flow to boundary conditions of the IR theory. It is a very interesting question, to which IR boundary condition a given boundary condition in the UV flows, or to formulate it in string theory terminology, what happens to a D-brane under closed string tachyon condensation.

If in the UV one starts with a conformal field theory defined on a surface Σ with boundary and a conformal boundary condition along $\partial\Sigma$, then, besides the regularisation already present in the bulk case, one also has to restrict the domains of the integrals (2.2) away from the boundary. This is due to a non-trivial singular bulk-boundary operator product expansion. Thus, in this situation there are two independent regularisation parameters, one of which parametrises the RG flow in the bulk, whereas the other one parametrises the induced flow in the boundary sectors. While these two flows are often treated simultaneously, we propose to perform the bulk flow first, while keeping the boundary regularisation fixed. This is nothing but a bulk flow on the subdomain

$$U_\epsilon := \{z \in \Sigma \,|\, \mathrm{dist}(z, \partial\Sigma) \geq \epsilon\} \subset \Sigma \tag{2.3}$$

of all points on Σ whose distance from the boundary $\partial\Sigma$ is bigger than the boundary regularisation parameter ϵ. Hence, the endpoint of the pure bulk flow with fixed boundary regularisation parameter is the IR theory on U_ϵ separated by a conformal defect line from the UV theory defined on the neighbourhood $\Sigma - U_\epsilon$ of the boundary.

The second step, namely the flow in the boundary sector is then described by letting ϵ go to zero, and in that way bringing the defect towards the boundary. This procedure produces out of the UV boundary condition a boundary condition of the IR theory. However, a priori this is a singular process, because the defect is a non-topological defect in general. That is not surprising. After all, the singularities appearing in the correlation functions when the perturbing fields φ_i come close to the boundary have not been cancelled by counterterms, because we have not performed the boundary RG flow. In fact, all the singularities arising at the boundary due to the entire bulk flow are encoded in the defect, and the process of taking the defect to the boundary has to be regularised in an appropriate way to obtain the induced flow in the boundary sectors.

Note that the approach we propose to describe the effect of bulk perturbation on boundary conditions is non-perturbative in the bulk coupling constants. Namely, the defect we associate to a bulk perturbation connects directly UV and IR theories of the corresponding RG flow. This obviously is an advantage, at least in the case where one can make sense of the procedure of bringing the defect close to the boundary. On the other hand, it is often not obvious how to connect perturbative and non-perturbative descriptions, i.e. how to find the defect associated to a particular bulk perturbation.

The structure of fusion of non-topological defects with boundaries is a very interesting subject, and has been considered for the case of the free boson in [25]. In this paper however we will avoid all intricacies arising in this context by considering $N = 2$ superconformal field theories. These theories can be topologically twisted, which in particular makes all defects preserving the appropriate supersymmetries topological. That means that they can be shifted on the surface without changing correlation functions, and in particular without

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giving rise to singularities when brought near boundaries or other defects. In this way there is a well defined fusion of supersymmetry preserving defects with supersymmetric boundary conditions or defects.

Our purpose in the following is to identify the supersymmetric defects associated to supersymmetry preserving perturbations of $N = 2$ superconformal field theories in the way described above. Their fusion with supersymmetric boundary conditions then describes to which boundary conditions the latter flow in the IR.

There are two kinds of supersymmetry preserving perturbations one can consider in $N = 2$ supersymmetric theories. Firstly there are pertrubations of the bulk action by chiral superfields Φ integrated over the chiral half of superspace[2]

$$\Delta S_c = \int_\Sigma d^2x\, d\theta^- d\theta^+\ \Phi|_{\bar{\theta}^\pm=0} + c.c.\,. \tag{2.4}$$

By construction, on surfaces without boundaries this perturbation leaves supersymmetry unbroken. On surfaces with boundaries however this is no longer the case, and the variation of the action gives rise to a boundary term [27, 26]. In case of an A-type boundary, this term can be compensated by the supersymmetry variation of a boundary integral of the form

$$\mathcal{B} = i \int_{\partial\Sigma} ds(\phi - \bar{\phi}) \tag{2.5}$$

which can be added to the action in order to preserve supersymmetry. If the boundary is of B-type, the boundary term cannot in general be cancelled in this manner, and perturbations of type (2.4) induce non-supersymmetric boundary flows.

The second type of supersymmetric bulk perturbation is given by integrals

$$\Delta S_t = \int_\Sigma d^2x\, d\bar{\theta}^- d\theta^+\ \Psi|_{\bar{\theta}^+=\theta^-=0} + c.c. \tag{2.6}$$

of twisted chiral superfields Ψ. In agreement with mirror symmetry, the boundary terms resulting from varying the action can be cancelled for B-type boundaries, but not for A-type ones.

Here, we are interested in bulk flows which also preserve supersymmetry in the boundary sectors. Thus, we can consider either chiral perturbation in the presence of A-type boundary conditions or twisted chiral perturbations in the presence of B-type boundary conditions.[3] As discussed above, performing the bulk RG flow while keeping the boundary regularisation parameter ϵ fixed, one obtains the IR theory on a domain $U_\epsilon \subset \Sigma$ and the IR theory on the neighbourhood $\Sigma - U_\epsilon$ of the boundary $\partial\Sigma$, which are separated by a defect on ∂U_ϵ.

Turning the arguments above around shows that perturbations with chiral superfields on a domain U give rise to A-type defects, whereas perturbations with twisted chiral

[2] Conventions on superspace are taken from [26].

[3] In the context of non-linear sigma models with Kähler target space, this corresponds to perturbations of the complex structure in the presence of A-branes or perturbations of the Kähler structure in the presence of B-branes.

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superfields give rise to B-type defects. Therefore, to identify what happens to the respective boundary conditions under a bulk flow, one first has to identify the corresponding A- or B-type defect and then analyse its fusion with the boundary condition. As alluded to above, the latter requires regularisation, but one can use supersymmetry to avoid dealing with it explicitly. Since the flow is supersymmetric all along, one can consider it in the topologically twisted theory, in which fusion of defects and boundary conditions is non-singular. This permits to identify the flows of topological boundary conditions, which in the situation we will consider here is enough to conclude the flows of the correspoding supersymmetric boundary conditions in the full conformal field theories.

3. Setup and outline

In the following we will apply and test the method outlined in section 2 in the case of orbifolds $\mathcal{M}_{d-2}/\mathbb{Z}_d$ of $N = 2$ superconformal minimal models. These are well studied rational conformal field theories at central charge $c = 3(1 - \frac{2}{d})$. (A few details about them are collected in appendix A). We are interested in perturbations which preserve supersymmetry. As discussed in section 2 there are two types of such perturbations, chiral and twisted chiral ones. The chiral ones are generated by (c, c)-chiral primary fields[4] and the twisted chiral ones by (a, c)-chiral primary fields. The corresponding perturbations are marginal or relevant if these fields have conformal weights $\leq \frac{1}{2}$. As it turns out, in $\mathcal{M}_{d-2}/\mathbb{Z}_d$ the (c, c)-chiral ring is trivial. But there are (a, c)-fields, which can be used to perturb the theory. More precisely, for every $i = 0, \ldots d - 1$ there is an (a, c)-field Ψ_i of conformal weights $h_i = \overline{h}_i = \frac{i}{2d} < \frac{1}{2}$, which can be obtained by spectral flow from the unique Ramond ground state in the ith twisted sector of the theory. Ψ_0 is the identity field, but all the other ones generate relevant perturbations, which drive the theory in the IR to another minimal model orbifold with smaller d however.

To understand these renormalisation group flows, the Landau-Ginzburg realisation of the involved models is very useful. The minimal model $\mathcal{M}_{d-2}$ can be obtained as IR limit of an $N = 2$ Landau-Ginzburg model with a single chiral superfield X and superpotential $W = X^d$. The orbifold group $\mathbb{Z}_d$ acts in the Landau-Ginzburg model by multiplication of X by dth roots of unity, so that the orbifold model $\mathcal{M}_{d-2}/\mathbb{Z}_d$ can be obtained as IR limit of the corresponding Landau-Ginzburg orbifold.

The RG flows can now be formulated in the framework of gauged linear sigma models, analogous to the flows between affine orbifold models $\mathbb{C}/\mathbb{Z}_d$ considered in [3]. We will not describe this approach here. Instead, we will study the perturbation in the mirror representation. The mirror of a minimal model orbifold $\mathcal{M}_{d-2}/\mathbb{Z}_d$ is just the minimal model $\mathcal{M}_{d-2}$ itself (see appendix A), and the mirror of the (a, c)-fields Ψ_i are the fields corresponding to the monomials X^i in the superfield X of the associated Landau-Ginzburg model. Thus, on the mirror side, perturbations generated by the Ψ_i are described by a Landau-Ginzburg model with superpotential $W = X^d$ deformed by lower order terms. Not being homogeneous anymore, the deformed superpotential effectively flows under the

[4] We are only interested in unitary flows, so that we always perturb with a conjugation invariant operator.

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renormalisation group due to field redefinitions (see e.g. [3]). It flows to a homogeneous superpotential corresponding to another conformal field theory.

As an example consider the perturbation of the orbifold model by the field $\Psi_{d'}$. On the mirror side this corresponds to deforming the superpotential $W = X^d$ to $W = X^d + \lambda X^{d'}$. The RG flow has the effect of scaling the superpotential as $W \mapsto \Lambda^{-1} W$. Accompanied by a field redefinition $X \mapsto \Lambda^{\frac{1}{d}} X$, this yields a running coupling constant $\lambda(\Lambda) = \lambda_0 \Lambda^{\frac{d'-d}{d}}$. In the UV ($\Lambda \to \infty$) the coupling goes to zero, whereas in the IR ($\Lambda \to 0$), the coupling diverges. Hence, this describes a flow between the Landau-Ginzburg models with superpotentials $W = X^d$ in the UV and the one with $W = X^{d'}$ in the IR. Therefore, the relevant operator $\Psi_{d'}$ induces an RG flow between the orbifolds $\mathcal{M}_{d-2}/\mathbb{Z}_d$ and $\mathcal{M}_{d'-2}/\mathbb{Z}_{d'}$.

Our aim is to study flows of this type, and in particular their effect on boundary conditions using defects between the minimal model orbifolds in UV and IR. Such defects cannot be topological, because the two conformal field theories are connected by a relevant flow and hence have different central charge. The construction of non-topological defects between conformal field theories is difficult, but since the flows preserve supersymmetry also the defects have to be supersymmetric. More precisely, being generated by twisted chiral fields, the flows give rise to B-type defects (c.f. section 2). Thus we can make use of a nice description of B-type defects between Landau-Ginzburg models in terms of matrix factorisations of the the difference of the respective superpotentials [18]. This formalism not only lends itself easily to the construction of defects, but also to the analysis of their fusion and their fusion with B-type boundary conditions, which are also represented in terms of matrix factorisations [28 − 31].

To be applicable to the study of defects between minimal model orbifolds, it has to be generalised to orbifolds of Landau-Ginzburg models. Similarly to B-type boundary conditions [32, 33], also B-type defects between Landau-Ginzburg orbifolds are described by matrix factorisations which are equivariant with respect to the action of the orbifold group, and properties like fusion generalise in a straight forward manner. This is discussed in section 4.

With this formalism at hand, in section 5 we construct a class of defects between minimal model orbifolds $\mathcal{M}_{d-2}/\mathbb{Z}_d$ with different d which we propose to arise in renormalisation group flows between these theories. This class of defects closes under fusion, i.e. the fusion of such a defect between minimal model orbifolds $\mathcal{M}_{d-2}/\mathbb{Z}_d$ and $\mathcal{M}_{d'-2}/\mathbb{Z}_{d'}$ and one between $\mathcal{M}_{d'-2}/\mathbb{Z}_{d'}$ and $\mathcal{M}_{d''-2}/\mathbb{Z}_{d''}$ is a defect between $\mathcal{M}_{d-2}/\mathbb{Z}_d$ and $\mathcal{M}_{d''-2}/\mathbb{Z}_{d''}$ of the same type. This fusion of defects indeed corresponds to the concatenation of renormalisation group flows.

As alluded to above, the flows we are interested in can be very explicitly studied in the mirror Landau-Ginzburg models, where they are just given by deformations of the superpotential. In particular, this can be used to investigate what happens to B-type boundary conditions in $\mathcal{M}_{d-2}/\mathbb{Z}_d$ under the flows. Namely, the corresponding mirror A-type boundary conditions have a very nice geometric interpretation as Lefshetz pencils of the superpotential W [27], whose behaviour under deformations of W can be determined explicitly. In section 6 we compare these flows to the fusion of our special defects with B-type boundary conditions calculated in section 5 and find complete agreement. This

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provides strong evidence for our claim that the special defects indeed are the ones which arise in the renormalisation group flows between different minimal model orbifolds.

As alluded to above, the flows between minimal model orbifolds $\mathcal{M}_{d-2}/\mathbb{Z}_d \mapsto \mathcal{M}_{d'-2}/\mathbb{Z}_{d'}$ studied here are very similar to flows between affine orbifold models $\mathbb{C}/\mathbb{Z}_d \mapsto \mathbb{C}/\mathbb{Z}_{d'}$. In section 7 we argue that our special defects between $\mathcal{M}_{d-2}/\mathbb{Z}_d$ have counterparts in the affine orbifolds $\mathbb{C}/\mathbb{Z}_d$, which describe the corresponding flows there.

4. B-type defects in Landau-Ginzburg orbifolds

Although the method described in section 2 above can be applied to study supersymmetric bulk flows of any theory with $N = 2$ supersymmetry, we will restrict our further discussion to (a, c)-perturbations of Landau-Ginzburg orbifold models. These can be described by means of B-type defects, and they remain supersymmetric on surfaces with boundary as long as B-type boundary conditions are imposed. B-type boundary conditions for Landau-Ginzburg models of chiral superfields X_i can be represented by matrix factorisations of the superpotential $W(X_i)$ [28–31].

In [18], see [16, 17, 34] for earlier work in a different context, it was shown that likewise B-type defects between two Landau-Ginzburg models with chiral superfields X_i and Y_i and superpotentials $W_1(X_i)$ and $W_2(Y_i)$ respectively can be described by means of matrix factorisations

$$\begin{aligned} P: \quad & P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0\,, \\ p_1 p_0 \;=\; & (W_1(X_i) - W_2(Y_i))\mathrm{id}_{P_0}\,, \quad p_0 p_1 = (W_1(X_i) - W_2(Y_i))\mathrm{id}_{P_1}\,. \end{aligned} \tag{4.1}$$

of the difference $W_1(X_i) - W_2(Y_i)$ of superpotentials. Here p_i are homomorphisms between the free $S = \mathbb{C}[X_i, Y_i]$-modules P_i.

In this section, we will extend the methods developed for Landau-Ginzburg defects to the case of Landau-Ginzburg orbifolds, in which defects can be represented by equivariant matrix factorisations. In particular fusion of defects and of defects with boundary conditions will be formulated in this formalism.

4.1 Defects and equivariant matrix factorisations

In the same way as for boundary conditions [32, 33] the matrix factorisation formalism for defects can be generalised to orbifolds of Landau-Ginzburg models. Namely, let Γ_1 and Γ_2 be orbifold groups of the respective LG models, i.e. Γ_1 acts on $\mathbb{C}[X_i]$ and Γ_2 on $\mathbb{C}[Y_i]$ in a way compatible with multiplication in these rings, such that $W_1(X_i)$ and $W_2(Y_i)$ are invariant.

A defect between the respective Landau-Ginzburg orbifolds is then given by a $\Gamma := \Gamma_1 \times \Gamma_2$-equivariant matrix factorisation of $W_1(X_i) - W_2(Y_i)$. The latter is a matrix factorisation (4.1) together with representations ρ_i of Γ on the modules P_i which are compatible with the S-module structure and with the maps p_i. That means that the Γ-action on the P_i defined by ρ_i satisfies

$$\rho_i(\gamma)(s \cdot p) = \rho(\gamma)(s) \cdot \rho_i(\gamma)(p)\,, \quad \forall \gamma \in \Gamma\,,\; s \in S\,,\; p \in P_i\,, \tag{4.2}$$

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where ρ denotes the action of Γ on S, and that furthermore the maps p_i commute with the actions ρ_i:

$$\rho_0(\gamma)p_1 = p_1\rho_1(\gamma)\,, \quad \rho_1(\gamma)p_0 = p_0\rho_0(\gamma)\,, \quad \forall\gamma\in\Gamma\,. \tag{4.3}$$

More details on equivariant matrix factorisations can be found e.g. in [32, 33].

In the cases we are interested in here, the orbifold groups are commutative. In particular, their action give the polynomial rings S the structure of graded rings, and the representations ρ_i turn the P_i into graded S-modules. Compatibility furthermore ensures that the maps p_i respect the grading, i.e. they are graded of degree 0. Matrix factorisations which are equivariant with respect to an abelian group action are therefore sometimes referred to as graded matrix factorisations.

Note that not all matrix factorisations P admit such representations ρ_i. Since the P_i are free S-modules, compatibility with the S-action is easily achieved, but the compatibility with the homomorphisms p_i is a non-trivial constraint. However, there is a standard procedure to construct from any matrix factorisation P a Γ-equivariant one by the orbifold construction, known for instance from the construction of boundary conditions in general orbifold theories from boundary conditions in the underlying non-orbifolded models. Given any matrix factorisation P, one considers the normal subgroup $\Gamma'\subset\Gamma$, which stabilises the matrix[5] p_1, hence also p_0, up to change of basis. Then one chooses a representation of Γ' on P, and extends it to a Γ-representation of the matrix factorisation given by the sum of the Γ/Γ'-orbit[6] of P

$$\widetilde{p}_i := \bigoplus_{\gamma\in\Gamma/\Gamma'}\gamma(p_i)\,, \qquad \widetilde{P}_i = \mathbb{C}[\Gamma/\Gamma']\otimes P_i\,. \tag{4.4}$$

This obviously defines an equivariant matrix factorisation of $W_1(X_i) - W_2(Y_i)$.

Given two Γ-equivariant matrix factorisations, the compatibility properties of the representations ρ_i ensure that the Γ-action lifts to an action on the corresponding BRST-cohomology groups of the matrix factorisations. The BRST-cohomology groups in the orbifold theories are then given by the Γ-invariant subgroups of the BRST-cohomology groups of the underlying matrix factorisations:

$$\mathcal{H}^*_{\mathrm{orb}}(P,Q) = \left(\mathcal{H}^*(P,Q)\right)^{\Gamma}\,. \tag{4.5}$$

4.2 Fusion

The most important property of defects which we will use is their fusion. The fusion of B-type defects in Landau-Ginzburg models has been discussed in [18]. Let us consider the situation of three LG models with chiral superfields X_i, Y_i, Z_i and superpotentials $W_1(X_i)$, $W_2(Y_i)$ and $W_3(Z_i)$ respectively. A B-type defect between the first two of these models can be fused with a B-type defect of the last two giving rise to a B-type defect between the first and the last of these models. On the level of matrix factorisation this can be described as follows. The first of these defects can be represented by a matrix factorisation P of $W_1(X_i)-$

[5] The p_i are matrices with entries in S, on which Γ acts.

[6] The group Γ/Γ' acts on the matrix factorisations stabilised by Γ'.

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$W_2(Y_i)$, whereas the second one is described by a matrix factorisation Q of $W_2(Y_i)-W_3(Z_i)$. The defect obtained by fusion of the two can now be represented by the matrix factorisation $P * Q$, which is defined to be the tensor product matrix factorisation $P \otimes Q$

$$(P\otimes Q)_1 = (P_1\otimes Q_0)\oplus(P_0\otimes Q_1) \underset{r_0}{\overset{r_1}{\rightleftarrows}} (P_0\otimes Q_0)\oplus(P_1\otimes Q_1) = (P\otimes Q)_0$$

$$\text{with} \quad r_1 = \begin{pmatrix} p_1\otimes \mathrm{id}_{Q_0} & -\mathrm{id}_{P_0}\otimes q_1 \\ \mathrm{id}_{P_1}\otimes q_0 & p_0\otimes \mathrm{id}_{Q_1} \end{pmatrix}, \quad r_0 = \begin{pmatrix} p_0\otimes \mathrm{id}_{Q_0} & \mathrm{id}_{P_1}\otimes q_1 \\ -\mathrm{id}_{P_0}\otimes q_0 & p_1\otimes \mathrm{id}_{Q_1} \end{pmatrix}, \tag{4.6}$$

regarded as a matrix factorisation over $S' = \mathbb{C}[X_i, Z_i]$. Here, by abuse of notation the tensor products between P_i and Q_j denote the tensor products $P_i \otimes_{\mathbb{C}[X_i,Y_i]} \mathbb{C}[X_i,Y_i,Z_i] \otimes_{\mathbb{C}[Y_i,Z_i]} Q_j$. Obviously, the matrix factorisation $P\otimes Q$ is a matrix factorisation of $W_1(X_i) - W_2(Y_i) + W_2(Y_i) - W_3(Z_i) = W_1(X_i) - W_3(Z_i)$. The bulk fields Y_i of the theory squeezed in between the original defects give rise to new defect degrees of freedom. Hence, $P * Q$ represents a defect between the LG models with superpotentials $W_1(X_i)$ and $W_3(Z_i)$ which a priori can however be of infinite rank. This happens, because the tensor product will in general still contain the variables Y_i. Interpreting $\mathbb{C}[X_i,Y_i,Z_i]$ as an infinite dimensional $\mathbb{C}[X_i,Z_i]$ module gives infinite rank to matrices that contain Y_i. As was shown in [18], the factorisations $P*Q$ are always equivalent to finite rank factorisations, provided P and Q are of finite rank.

The generalisation of the matrix factorisation representation of fusion to Landau-Ginzburg orbifolds is straight forward. The same arguments as in non-orbifolded Landau-Ginzburg models leads one to consider the tensor product matrix factorisation $P\otimes Q$. But now, P and Q are equivariant with respect to $\Gamma_1\times\Gamma_2$ and $\Gamma_2\times\Gamma_3$ respectively. Thus, $P\otimes Q$ is equivariant with respect to $\Gamma_1\times\Gamma_2\times\Gamma_3$. Again $P\otimes Q$ has to be regarded as ($\Gamma_1\times\Gamma_3$ equivariant) matrix factorisation over $\mathbb{C}[X_i,Z_i]$, because the Y_i become new defect degrees of freedom. Just like for the BRST-cohomology, the orbifold causes a projection onto Γ_2-invariant degrees of freedom. Thus

$$P *_{\mathrm{orb}} Q = (P*Q)^{\Gamma_2}. \tag{4.7}$$

This discussion easily extends to fusion of B-type defects with B-type boundary conditions. For this, one only has to replace the matrix factorisation Q above by a Γ_2-graded matrix factorisation of $W_2(Y_i)$ which represents a boundary condition in the LG model with superpotential $W_2(Y_i)$. $P *_{\mathrm{orb}} Q$ is then a Γ_1-equivariant matrix factorisation of $W_1(X_i)$ and represents a boundary condition in the corresponding LG orbifold.

4.3 Quantum symmetry defects in $X^d/\mathbb{Z}_d$

As an example let us discuss defects between one and the same Landau-Ginzburg orbifold with only one chiral superfield X, superpotential $W(X) = X^d$ and orbifold group $\Gamma = \mathbb{Z}_d$ acting on X by

$$X \mapsto \xi^a X, \quad a\in\mathbb{Z}_d, \tag{4.8}$$

where ξ is an elementary dth root of unity. A simple defect in the unorbifolded LG model is the identity defect which can be represented by the matrix factorisation [18]

$$P: \quad P_1 = S \underset{p_0=\prod_{i\neq 0}(X-\xi^i Y)}{\overset{p_1=(X-Y)}{\rightleftarrows}} S = P_0, \tag{4.9}$$

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with $S = \mathbb{C}[X, Y]$. To obtain out of this a $\Gamma \times \Gamma$-equivariant matrix factorisation, one can use the orbifold construction described above. The subgroup stabilising P is given by the diagonal subgroup $\Gamma_{\text{diag}} \subset \Gamma \times \Gamma$. Thus, the first step is to choose a $\Gamma_{\text{diag}} \cong \mathbb{Z}_d$ representation on P. This is done by specifying the $\mathbb{Z}_d$ representation m on the subspace spanned by $1 \in P_0 \cong S$, which by compatibility with the S-action extends to a representation on P_0, and which by compatibility with the maps p_i determines a representation on P_1. One obtains the Γ_{diag}-equivariant matrix factorisation

$$S[m+1] \underset{p_0=\prod_{i\neq 0}(X-\xi^i Y)}{\overset{p_1=(X-Y)}{\rightleftarrows}} S[m]\,. \tag{4.10}$$

The $\Gamma/\Gamma' \cong \{1\} \times \Gamma$-orbit of this matrix factorisation yields the $\Gamma \times \Gamma$-equivariant matrix factorisation (we only specify $\widetilde{p}_1$ here)

$$\widetilde{p}_1 = \bigoplus_{i\in\mathbb{Z}_d} (X - \xi^i Y) : (S[m+1])^{\oplus d} \to (S[m])^{\oplus d}\ . \tag{4.11}$$

The representation of $\Gamma \times \Gamma$ on $\widetilde{P}_0$ is determined by the representation $\overline{\rho}_0$ on the subspace $\overline{P}_0 = \mathbb{C}[m]^{\oplus d} \subset (S[m])^{\oplus d}$. It is given by

$$(a, b) \in \mathbb{Z}_d \times \mathbb{Z}_d : \quad \overline{\rho}_0(a, b) = \xi^{\text{am}} \text{id}_{\overline{P}_0} + \epsilon^{b-a}\,, \tag{4.12}$$

where ϵ is the $d \times d$-matrix defined by $\epsilon_{i,j} = \delta^{(d)}_{i-j-1}$. It is now easy to diagonalise $\overline{\rho}_i$ on $\overline{P}_i$. In the corresponding eigenbasis the equivariant matrix factorisation $\widetilde{P}^m(X, Y)$ reads

$$\widetilde{p}_1^m = \begin{pmatrix} X & & & -Y \\ -Y & \ddots & & \\ & \ddots & \ddots & \\ & & -Y & X \end{pmatrix} : S^d \begin{pmatrix} [m+1,0] \\ \vdots \\ [m+d,-d+1] \end{pmatrix} \longrightarrow S^d \begin{pmatrix} [m,0] \\ \vdots \\ [m+d-1,-d+1] \end{pmatrix}, \tag{4.13}$$

where now $[\cdot, \cdot]$ denotes an irreducible $\Gamma \times \Gamma$-representation defined on the respective subspaces of $\overline{P}_i$.

What we have seen in particular is that the identity defects P of the unorbifolded theory breaks up into d different defects $\widetilde{P}^m$ of the $\mathbb{Z}_d$ orbifold. One expects of course that one of them can be identified as the identity defect of the orbifold. We will now show that the $\widetilde{P}^m$ realise the $\mathbb{Z}_d$ group of quantum symmetries of the orbifold theory.

To see this, we first calculate the fusion of two such defects represented by matrix factorisations $P = \widetilde{P}^m(X, Y)$ and $Q = \widetilde{P}^{m'}(Y, Z)$. As discussed above, the result of the composition is given by $P *_{\text{orb}} Q$, the $\mathbb{Z}_d$-invariant part of the tensor product matrix factorisation $P \otimes Q$. Indeed, as in the unorbifolded situation, there is a trick, which simplifies the calculation of the fusion. Namely, the tensor product matrix factorisation is equivalent to the matrix factorisation arising after two steps out of a $\mathbb{C}[X, Z]$-free two-periodic resolution of the module [18]

$$M = \text{coker}(p_1 \otimes \text{id}_{Q_0}, \text{id}_{P_0} \otimes q_1)\,, \tag{4.14}$$

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and the $\mathbb{Z}_d$-invariant part is equivalent to the matrix factorisation arising in the same way out of $M^{\mathbb{Z}_d}$. In order to calculate $M^{\mathbb{Z}_d}$, let us first note that $P_0 \otimes Q_0$ is generated over $\widehat{S} = \mathbb{C}[X,Y,Z]$ by $e_{a,b} = e_a^P \otimes e_b^Q$ of Γ^3-degree $[m+a, -a+m'+b, -b]$, where $(e_a^P)_{a\in\mathbb{Z}_d}$ and $(e_b^Q)_{b\in\mathbb{Z}_d}$ are generating systems of P_0 and Q_0 respectively. Generators of $P_0 \otimes Q_0$ over $S' = \mathbb{C}[X,Z]$ are given by $e_{a,b}^i = Y^i e_{a,b}$. The relations in M coming from $p_1 \otimes \mathrm{id}_{Q_0}$ and $\mathrm{id}_{P_0} \otimes q_1$ in this basis read

$$e_{a+1,b}^{i+1} = X e_{a,b}^i\,, \qquad e_{a,b}^{i+1} = Z e_{a,b+1}^i \quad \forall i \geq 0\,. \tag{4.15}$$

The second of these relations can be used to eliminate all $e_{a,b}^i$ for $i > 0$ from the generating system of M. The remaining relations then become

$$X e_{a,b}^0 = Z e_{a+1,b+1}^0\,. \tag{4.16}$$

Hence M is generated by $e_{a,b}^0$ subject to these relations. Moreover, $M^{\mathbb{Z}_d}$ is generated by those generators, which are $\mathbb{Z}_d$-invariant (with respect to the second $\mathbb{Z}_d$), i.e. $f_a := e_{a,a-m'}^0$ subject to the relations

$$X f_a = Z f_{a+1}\,. \tag{4.17}$$

Moreover, the $\mathbb{Z}_d \times \mathbb{Z}_d$ degree of f_a is given by $[m+a, m'-a]$. Therefore

$$M^{\mathbb{Z}_d} = \mathrm{coker}(\widetilde{p}_1^{m+m'}(X,Z))\,, \tag{4.18}$$

and $M^{\mathbb{Z}_d}$ has a S'-free resolution given by $\widetilde{P}^{m+m'}(X,Z)$. Hence for the fusion we obtain

$$\widetilde{P}^m(X,Y) *_{\mathrm{orb}} \widetilde{P}^{m'}(Y,Z) = \widetilde{P}^{m+m'}(X,Z)\,. \tag{4.19}$$

Indeed the $\widetilde{P}^m$ form a $\mathbb{Z}_d$-group under fusion.

We will now check that the action of the defects on boundary conditions reproduces the action of the $\mathbb{Z}_d$ of quantum symmetries. Boundary conditions in this model are described by $\mathbb{Z}_d$-equivariant matrix factorisations of W. These can be decomposed into the irreducible ones

$$Q^{(M,N)}(X): \quad Q_1 = \mathbb{C}[X][M+N] \underset{q_0 = X^{d-N}}{\overset{q_1 = X^N}{\rightleftarrows}} Q_0 = \mathbb{C}[X][M]\,, \tag{4.20}$$

for $(M,N) \in \mathcal{I}_d = \mathbb{Z}_d \times \{0,\ldots,d-1\}$. The quantum symmetries act on these matrix factorisations by shifting the $\mathbb{Z}_d$-representation label M.

To calculate the fusion $P *_{\mathrm{orb}} Q$ for $P = \widetilde{P}^m(X,Y)$ with $Q = Q^{(M,N)}(Y)$ we follow the same path as before. The module $M = \mathrm{coker}(p_1 \otimes \mathrm{id}_{Q_0}, \mathrm{id}_{P_0} \otimes q_1)$ is generated over $S' = \mathbb{C}[X]$ by $e_a^i = Y^i e_a^P \otimes e^Q$ of $\mathbb{Z}_d \times \mathbb{Z}_d$-degree $[m+a, -a+M+i]$ with relations

$$e_{a+1}^{i+1} = X e_a^i\,, \qquad e_a^{N+i} = 0\,, \quad \forall i \geq 0\,. \tag{4.21}$$

The first set of relations can again be used to reduce the generating system to e_a^0, and the remaining relations are

$$X^N e_{a-N}^0 = 0\,. \tag{4.22}$$

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The only $\mathbb{Z}_d$-invariant generator is e^0_M, hence

$$M^{\mathbb{Z}_d} = \mathrm{coker}(q_1^{(m+M,N)}(X)) , \tag{4.23}$$

which therefore has an S'-free resolution given by $Q^{(m+M,N)}(X)$. We arrive at

$$\widetilde{P}^m(X,Y) *_{\mathrm{orb}} Q^{(M,N)}(Y) = Q^{(M+m,N)}(X) . \tag{4.24}$$

In particular the defects corresponding to $\widetilde{P}^m$ are the generators of the quantum $\mathbb{Z}_d$-symmetry in the LG orbifold, and the one for $m = 0$ is the identity defect.

The construction of these defects (and a more general class of topological defects) is indeed also straight forward on the level of conformal field theory. We have presented it in appendix A.

5. A special class of defects between $X^d/\mathbb{Z}_d$ and $X^{d'}/\mathbb{Z}_{d'}$

In the following we will focus our attention to orbifolds of Landau-Ginzburg models with one chiral superfield X and superpotential $W = X^d$ for some d. The orbifold group $\Gamma = \mathbb{Z}_d$ acts on X by multiplication with dth roots of unity.

In this section we will define a special class of B-type supersymmetric defects between such models by constructing a class of $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-graded matrix factorisations of $Y^{d'} - X^d$. We will then determine their fusion among themselves and their fusion with B-type boundary conditions.

5.1 Construction

The matrix factorisations defining the special defects are determined by irreducible representations m of $\mathbb{Z}_d$ and a d'-tuple of integers $n = (n_0, \ldots, n_{d'-1})$, $n_i \in \mathbb{N}_0$ such that $\sum_i n_i = d$. We will denote the set of all such pairs (m,n) by $\mathcal{I}_{d',d}$. Given an n as above, define the following $d' \times d'$-matrix

$$(\Xi_n)_{a,b} := \delta^{(d')}_{a,b+1} X^{n_a} . \tag{5.1}$$

This matrix has the property that

$$\Xi_n^{d'} = X^d \mathrm{id}_{d'} \tag{5.2}$$

and hence can be used to construct matrix factorisations of $Y^{d'} - X^d$ by means of

$$\left(Y^{d'} - X^d\right) \mathrm{id}_{d'} = \prod_{i=0}^{d'-1} \left(Y \mathrm{id}_{d'} - \xi^i \Xi_n\right) , \tag{5.3}$$

where ξ is an elementary d'th root of unity. In particular choosing a subset $I \subset \{0, \ldots, d'-1\}$ one obtains a matrix factorisation of $Y^{d'} - X^d$ by grouping together the corresponding factors into one matrix and the ones corresponding to the complement into the other:

$$p_1 = \prod_{i \in I} (Y \mathrm{id}_{d'} - \xi^i \Xi_n) , \quad p_0 = \prod_{i \in \{0,\ldots,d'-1\} - I} (Y \mathrm{id}_{d'} - \xi^i \Xi_n) . \tag{5.4}$$

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This matrix factorisation is $\mathbb{Z}_{d'} \times \mathbb{Z}_d$ gradable. The grading is determined by the grading of a single factor $(Y\mathrm{id}_{d'} - \xi^i \Xi_n)$. In particular, the grading of any matrix factorisation of type (5.4) can be obtained from the grading of the matrix factorisation with $I = \{0\}$ on which we will focus now. For $I = \{0\}$, given $(m,n) \in \mathcal{I}_{d',d}$, the respective graded matrix factorisation $P^{(m,n)} = P^{(m,n)}_{\{0\}}$ is defined by

$$p_1^{(m,n)} = (Y\mathrm{id}_{d'} - \Xi_n) = \begin{pmatrix} Y & & & -X^{n_0} \\ -X^{n_1} & \ddots & & \\ & \ddots & \ddots & \\ & & -X^{n_{d'-1}} & Y \end{pmatrix} : P_1 \longrightarrow P_0 \,, \tag{5.5}$$

where

$$P_1 = S^{d'} \begin{pmatrix} [1,-m] \\ [2,-m-n_1] \\ [3,-m-n_1-n_2] \\ \vdots \\ \left[d',-m-\sum_{i=1}^{d'-1} n_i\right] \end{pmatrix}, \qquad P_0 = S^{d'} \begin{pmatrix} [0,-m] \\ [1,-m-n_1] \\ [2,-m-n_1-n_2] \\ \vdots \\ \left[d'-1,-m-\sum_{i=1}^{d'-1} n_i\right] \end{pmatrix}. \tag{5.6}$$

Here $S = \mathbb{C}[X,Y]$, and $[\cdot,\cdot]$ denotes the $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-degree. Note that because we are in the orbifold category, symmetry operations $X \mapsto \eta^i X$, $Y \mapsto \xi^j Y$, where η is an elementary dth root of unity act trivially on the matrix factorisations above.

5.2 Fusion of defects

Let us consider two matrix factorisations of the type defined in (5.5). For $(m,n) \in \mathcal{I}_{d',d}$ and $(\widetilde{m},\widetilde{n}) \in \mathcal{I}_{d'',d'}$ let

$$P := P^{(m,n)}(Y,X)\,, \qquad Q := P^{(\widetilde{m},\widetilde{n})}(Z,Y) \tag{5.7}$$

be the respective graded matrix factorisations of $Y^{d'} - X^d$ and $Z^{d''} - Y^{d'}$ respectively. We would like to calculate the fusion of the respective defects. As discussed in section 4 the fused defect can be represented by the $\mathbb{Z}_{d'}$-invariant part of the tensor product matrix factorisation $P \otimes Q$ regarded as matrix factorisation over $S' := \mathbb{C}[X,Z]$. By the usual trick [18] which has already been used in the discussion of the fusion of the quantum symmetry defects in section 4.3 it can be obtained as the matrix factorisation associated to the $\mathbb{Z}_{d'}$-invariant part of the module

$$M = \mathrm{coker}(p_1 \otimes \mathrm{id}_{Q_0}, \mathrm{id}_{P_0} \otimes q_1)\,. \tag{5.8}$$

Here as in section 4.3 above, by abuse of notation $P_i \otimes Q_j$ denotes the tensor product over $\widehat{S} = \mathbb{C}[X,Y,Z]$ of the respective $\widehat{S}$-modules $P_i \otimes_{\mathbb{C}[X,Y]} \widehat{S}$ and $Q_i \otimes_{\mathbb{C}[Y,Z]} \widehat{S}$, and M is regarded as an S'-module.

In order to analyse M, let us denote by $(e^P_a)_{a\in\mathbb{Z}_{d'}}$ the free generators of P_0 of $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-degrees $[e^P_a] = [a, -m - \sum_{i=1}^{a} n_i]$, and by $(e^Q_b)_{b\in\mathbb{Z}_{d''}}$ the generators of Q_0 with $\mathbb{Z}_{d''} \times \mathbb{Z}_{d'}$-degree $[e^Q_b] = [b, -\widetilde{m} - \sum_{i=1}^{b} \widetilde{n}_i]$. We define the corresponding generators $e_{a,b} := e^P_a \otimes e^Q_b$

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of $P_0 \otimes Q_0$ of $\mathbb{Z}_{d''} \times \mathbb{Z}_{d'} \times \mathbb{Z}_d$-degree

$$[e_{a,b}] = \left[b, a - \widetilde{m} - \sum_{i=1}^{b} \widetilde{n}_i, -m - \sum_{i=1}^{a} n_i \right] . \tag{5.9}$$

As an $S' = \mathbb{C}[X, Z]$-module, $P_0 \otimes Q_0$ is generated by $e^j_{a,b} := Y^j e_{a,b}$. In this basis the relations in M coming from $p_1 \otimes \mathrm{id}_{Q_0}$ can be written as

$$e^{j+1}_{a,b} = X^{n_{a+1}} e^j_{a+1,b} \quad \forall j \in \mathbb{N}_0 . \tag{5.10}$$

They imply

$$e^j_{a,b} = X^{\sum_{i=1}^{j} n_{a+i}} e^0_{a+j,b} , \tag{5.11}$$

and can hence be used to eliminate $e^j_{a,b}$ with $j > 0$ from the generating system of M.

The relations coming from $\mathrm{id}_{P_0} \otimes q_1$ on the other hand read

$$Z e^j_{a,b} = e^{j+\widetilde{n}_{b+1}}_{a,b+1} \quad \forall j \in \mathbb{N}_0 . \tag{5.12}$$

Using (5.11) they become

$$Z X^{\sum_{i=1}^{j} n_{a+i}} e^0_{a+j,b} = X^{\sum_{i=1}^{j+\widetilde{n}_{b+1}} n_{a+i}} e^0_{a+j+\widetilde{n}_{b+1},b+1} \quad \forall j \in \mathbb{N}_0 . \tag{5.13}$$

Obviously, the relations (5.13) for $j > 0$ follow from the ones with $j = 0$, so that M is isomorphic to the S'-module generated by $e^0_{a,b}$ subject to the relations

$$Z e^0_{a,b} = X^{\sum_{i=1}^{\widetilde{n}_{b+1}} n_{a+i}} e^0_{a+\widetilde{n}_{b+1},b+1} . \tag{5.14}$$

In particular, $M^{\mathbb{Z}_{d'}}$ is isomorphic to the S'-module generated by the $\mathbb{Z}_{d'}$-invariant generators

$$f_b := e_{\widetilde{m}+\sum_{j=1}^{b} \widetilde{n}_j, b} \tag{5.15}$$

subject to the relations

$$Z f_b = X^{\sum_{i=1}^{\widetilde{n}_{b+1}} n_{\widetilde{m}+\sum_{j=1}^{b} \widetilde{n}_j + i}} f_{b+1} . \tag{5.16}$$

These relations can indeed be represented by a matrix of type (5.5). More precisely

$$M^{\mathbb{Z}_{d'}} \cong \mathrm{coker}\left(p_1^{(\widehat{m},\widehat{n})}(X, Z) \right) \tag{5.17}$$

with $(\widehat{m}, \widehat{n}) \in \mathcal{I}_{d'',d}$ given by

$$\widehat{m} = m + \sum_{i=1}^{\widetilde{m}} n_i , \quad \widehat{n}_{b+1} = \sum_{i=1}^{\widetilde{n}_{b+1}} n_{\widetilde{m}+\sum_{j=1}^{b} \widetilde{n}_j + i} . \tag{5.18}$$

Therefore, the class of defects defined by matrix factorisations (5.5) is closed under fusion. For every $(m, n) \in \mathcal{I}_{d',d}$ and $(\widetilde{m}, \widetilde{n}) \in \mathcal{I}_{d'',d'}$ fusion is given by

$$P^{(\widetilde{m},\widetilde{n})} * P^{(m,n)} = P^{(\widehat{m},\widehat{n})} , \tag{5.19}$$

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where $(\widehat{m},\widehat{n}) =: (\widetilde{m},\widetilde{n}) * (m,n) \in \mathcal{I}_{d'',d}$ is defined by (5.18). Indeed, it is not difficult to see that for every $(\widehat{m},\widehat{n}) \in \mathcal{I}_{d',d}$ there exist $(m_i,n_i) \in \mathcal{I}_{d+i+1,d+i}$, $0 \leq i < d'-d$ such that

$$(\widehat{m},\widehat{n}) = (m_{d'-d}, n_{d'-d}) * \ldots * (m_0,n_0). \tag{5.20}$$

That means that every defect $P^{(\widehat{m},\widehat{n})}$ between Landau-Ginzburg orbifolds $X^d/\mathbb{Z}_d$ and $X^{d'}/\mathbb{Z}_{d'}$ can be obtained by fusion of $|d'-d|$ defects between Landau-Ginzburg orbifolds with $|d-d'|=1$. This will become more evident using a pictorial representation of these defects which will be introduced in section 5.4 below after the discussion of their action on boundary conditions.

It is also easy to calculate the fusion of the defects $P^{(m,n)}$ with the defects $\widetilde{P}^m$ representing the quantum symmetries. One obtains

$$\widetilde{P}^{m''} * P^{(m,n)} * \widetilde{P}^{m'} = P^{(\widehat{m},\widehat{n})}, \tag{5.21}$$
$$\widehat{m} = m + m' + \sum_{j=1}^{\{-m''\}_{d'}} n_j\,, \quad \widehat{n} = (\widehat{n}_0,\ldots,\widehat{n}_{d'}) = (n_{-m''}, n_{-m''+1},\ldots,n_{d'-m''-1})\,.$$

5.3 Fusion of defects and boundary conditions

Next, we would like to calculate what happens to B-type boundary conditions in Landau-Ginzburg orbifolds $X^d/\mathbb{Z}_d$ upon fusion with a defect represented by $P^{(m,n)}$ for some $(m,n) \in \mathcal{I}_{d',d}$. B-type boundary conditions in this model can be represented by $\mathbb{Z}_d$-graded matrix factorisations of X^d. As already mentioned in section 4, the latter can be decomposed into sums of the irreducible matrix factorisations

$$Q^{(M,N)}: \quad Q_1 = \mathbb{C}[X][M+N] \underset{q_0 = X^{d-N}}{\overset{q_1 = X^N}{\rightleftarrows}} Q_0 = \mathbb{C}[X][M]\,, \tag{5.22}$$

for $(M,N) \in \mathcal{I}_d = \mathbb{Z}_d \times \{0,\ldots,d-1\}$. Thus, it is sufficient to determine the fusion of $P^{(m,n)}$ with these.

Similar to the case of fusion of defects also the boundary condition created by fusing the defect associated to $P = P^{(m,n)}$ with the boundary condition associated to $Q = Q^{(M,N)}$ is represented by the matrix factorisation obtained from the $\mathbb{Z}_d$-invariant submodule of

$$M = \mathrm{coker}(p_1 \otimes \mathrm{id}_{Q_0}, \mathrm{id}_{P_0} \otimes q_1) \tag{5.23}$$

regarded as $S' = \mathbb{C}[Y]$-module. We denote the $S = \mathbb{C}[X,Y]$-free generators of $P_0 \otimes Q_0$ by e_a, $a \in \mathbb{Z}_{d'}$. They have $\mathbb{Z}_{d'} \times \mathbb{Z}_d$-degree $[a, -m - \sum_{i=1}^a n_i + M]$. S'-free generators of $P_0 \otimes Q_0$ are given by $e_a^i = X^i e_a$, $i \geq 0$. In this generators, the relations in M can be written as

$$Y e_a^i = e_{a+1}^{i+n_{a+1}}\,, \quad e_a^{N+i} = 0\,. \tag{5.24}$$

By means of these relations, one can reduce the set of generators to those e_a^i with $0 \leq i \leq \min(N, n_a) - 1$. The $\mathbb{Z}_d$-invariant ones are the ones with

$$i = i(a) = \left\{ m - M + \sum_{j=1}^{a} n_j \right\}_d\,, \tag{5.25}$$

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where $\{z\}_d$ denotes the representative in $\mathbb{Z}$ of $z \in \mathbb{Z}_d$ which lies in $[0, d-1]$. A generator $e_a^{i(a)}$ contributes to $M^{\mathbb{Z}_d}$ iff $i(a) < \min(N, n_a)$. Using the relation

$$Y^k e_a^i = e_{a+k}^{i+\sum_{j=1}^k n_{a+j}} \tag{5.26}$$

one easily obtains that $e_a^{i(a)}$ generates a submodule with relation

$$Y^k e_a^{i(a)} = 0\,, \quad \forall k : \, i(a) + \sum_{j=1}^{k} n_{a+j} \geq N\,. \tag{5.27}$$

The $\mathbb{Z}_{d'}$-degree of this generator is given by $[e_a^{i(a)}] = [a]$. Hence

$$M^{\mathbb{Z}_d} \cong \bigoplus_{a \in \mathbb{Z}_{d'} \,:\, i(a) = \{m - M + \sum_{j=1}^{a} n_j\}_d < \min(N, n_a)} \operatorname{coker}\left(q_1^{(a,k(a))}\right)\,, \tag{5.28}$$

where

$$k(a) = \min\left\{ j > 0 \,|\, i(a) + \sum_{k=1}^{j} n_{a+k} \geq N \right\}\,. \tag{5.29}$$

In particular, the fusion reads

$$P^{(m,n)} * Q^{(M,N)} = \bigoplus_{a \in \mathbb{Z}_{d'} \,:\, i(a) = \{m - M + \sum_{j=1}^{a} n_j\}_d < \min(N, n_a)} Q^{(a,k(a))}\,. \tag{5.30}$$

Indeed, for all $(m, n) \in \mathcal{I}_{d',d}$ and $(M, N) \in \mathcal{I}_d$ this sum has at most one summand. This can easily be seen as follows. Suppose $i(a) < n_a$ for some $a \in \mathbb{Z}_{d'}$, giving rise to a possible summand in (5.30). Then

$$i(a') = i(a) + \sum_{j=1}^{\{a'-a\}_{d'}} n_{a+j}\,, \tag{5.31}$$

because from $i(a) < n_a$ it follows that the right hand side is $< d$ for all a'. Since now all the summands are non-negative this implies that $i(a') \geq n_{a'}$ for all $a' \neq a$, and therefore no $a' \neq a$ can contribute to the sum in (5.30).

However, the sum in (5.30) can be empty if there exist $n_i \geq 2$. More precisely, for each $n_i \geq 2$ matrix factorisations $Q = Q^{(M,N)}$ are annihilated[7] by $P^{(m,n)}$, iff

$$N \leq n_i - 1\,, M \in (m + 1 + n_1 + \cdots + n_{i-1}) + \{0, \ldots, n_i - N - 1\}\,. \tag{5.32}$$

This can be seen by considering the set $\mathcal{J} := \{i(a) \,|\, a \in \mathbb{Z}_{d'}\}$ of possible values of $i(a)$. $\mathcal{J}$ is a subset of $\mathbb{Z}_d$, and its complement is given by

$$\mathcal{J}^c = (m - M + 1) + \Big([0, n_1 - 2] \cup (n_1 + [0, n_2 - 2]) \cup \ldots \cup (n_1 + \cdots + n_{d-1} + [0, n_d - 2])\Big)\,. \tag{5.33}$$

[7] Note that since the supersymmetric boundary conditions in the models at hand are classified, one immediately obtains that the fusion of the respective defects and boundary conditions also vanishes identically in the full conformal field theory, provided this fusion is regularised in a supersymmetric way.

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In particular for $M = m+1+n_1+\cdots+n_{i-1}+r$, $0 \le r \le n_i - N - 1$

$$\mathcal{J}^c = (-n_1 - \cdots - n_{i-1} - r + [0, n_1 - 2]) \cup \ldots \cup (-r + [0, n_i - 2]) \cup \\ \cup \ldots \cup (-r + n_i + \cdots + n_{d-1} + [0, n_d - 2]) \,. \tag{5.34}$$

But this means that for all $a \in \mathbb{Z}_{d'}$ $i(a) > n_i - 2 - r \ge N - 1$, i.e. $i(a) \ge N$ for all a, and hence the sum in (5.30) is empty.

Let us suppose now, that $Q^{(M,N)}$ is not annihilated. This means that

$$\mathcal{J} \cap \{0, \ldots, N-1\} = \{i_1, \ldots, i_l\} \neq \emptyset \,. \tag{5.35}$$

Assume $i_1 = i(a_1)$ is the smallest of the i_j (considered as elements of $\mathbb{Z}$ in the range $\{0, \ldots, d-1\}$). Then, $i_1 < n_{a_1}$, because otherwise $0 \le i_1 - n_{a_1} < i_1$ would also be an element of the set above. Hence

$$P^{(m,n)} * Q^{(M,N)} = Q^{(a_1, k(a_1))} \,. \tag{5.36}$$

This formula looks rather implicit, but there is a nice pictorial way to understand it, which we will discuss in the next section.

5.4 Pictorial representation of defect action

Let us for the moment restrict the discussion to those $P = P^{(m,n)}$ with $n_i \ge 1$ for all i. This implies in particular $d \ge d'$. Obviously this property is preserved under fusion. The first thing to note is that under this assumption the action of P on a matrix factorisation $Q^{(M,N)}$ does not increase N, which is obvious from (5.29). This implies in particular that under the fusion with P, $Q^{(M,1)}$ is either annihilated or it is mapped to $Q^{(M',1)}$ for some $M' \in \mathbb{Z}_{d'}$. The ones which are not annihilated are the ones such that there exists an $a \in \mathbb{Z}_{d'}$ with $i(a) = 0$, i.e. those with

$$M \in m + \{0, n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{d'-1}\} =: \mathcal{L}_{(m,n)} \,, \tag{5.37}$$

and for $M = m + \sum_{j=1}^{a} n_j$ one obtains $M' = a$. Summarising, for the action of P on the $N = 1$ boundary conditions we get

$$P^{(m,n)} * Q^{(M,1)} = \begin{cases} 0\,, & \text{if } M \notin m + \{\sum_{i=1}^{a} n_i \,|\, 0 \le a < d'\} \\ Q^{(a,1)}\,, & \text{if } M = m + \sum_{i=1}^{a} n_i \end{cases} \,. \tag{5.38}$$

This suggest the following picture for the action of the defects P on the $Q^{(M,1)}$. Consider a disk subdivided by straight lines from its center to its boundary into d sectors. Mark one of the straight lines, and denote the sectors by S_0 to S_{d-1} going in counterclockwise direction and starting from the marked line (c.f. figure 2a). In this picture we represent matrix factorisations $Q^{(M,1)}$ by the Mth sector S_M.

Now we can consider the following pictorial operations. The first rather trivial one $\mathcal{T}_m$ is the shift of the marking to the $-m$th line in counterclockwise direction, which just corresponds to the quantum symmetry $Q^{(M,1)} \mapsto Q^{(M+m,1)}$ (c.f. figure 3a). A more interesting

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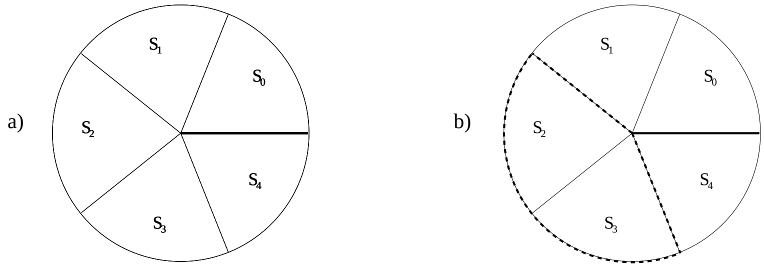

Figure 2: a) Disk subdivided into d sectors S_i representing boundary conditions $Q^{(i,1)}$. b) Union of consecutive sectors represent boundary conditions $Q^{(M,N)}$ with $N > 1$, e.g. $S_2 \cup S_3$ representing boundary condition $Q^{(2,2)}$.

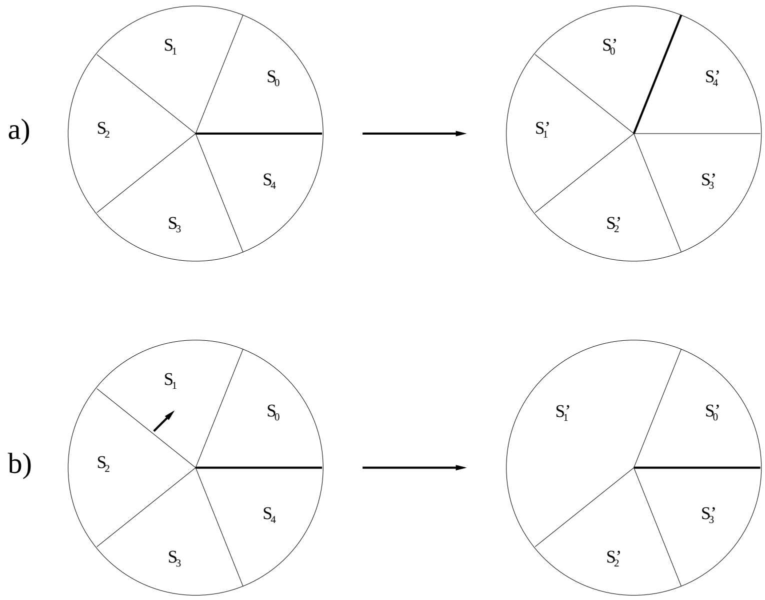

Figure 3: Diskoperations: a) $\mathcal{T}_{-1}$: marked line shifted by 1, $S_i \mapsto S'_{i-1}$, b) $\mathcal{S}_{\{1\}}$: sector S_1 shrunken to zero, $S_0 \mapsto S'_0$, $S_1 \mapsto 0$, $S_i \mapsto S'_{i-1}$ for $1 < i \leq 4$.

operation is the operation $\mathcal{S}_{\{s_1,\ldots,s_{d-d'}\}}$, which shrinks to zero the sectors S_{s_i} by bringing together the lines bounding them. In this way, from a disk subdivided into d sectors S_M one obtains a disk subdivided into d' sectors $S'_{M'}$ again counted in counterclockwise direction from the marked line (c.f. figure 3b). By means of the identification of boundary conditions $Q^{(M,1)}$ with sectors S_M the operation of $P^{(m,n)}$ in (5.38) can be written as

$$\mathcal{O}^{(m,n)} = \mathcal{S}_{\mathcal{L}^c_{(m,n)}-m}\mathcal{T}_{-m} = \mathcal{T}_{-a_{(m,n)}}\mathcal{S}_{\mathcal{L}^c_{(m,n)}}\,, \tag{5.39}$$

where $\mathcal{L}^c_{(m,n)}$ is the complement of the set $\mathcal{L}_{(m,n)}$ of $\mathbb{Z}_d$-labels of the non-annihilated $N = 1$-boundary conditions. $a_{(m,n)} := |\{0,\ldots,m\} \cap \mathcal{L}_{(m,n)}|$ is the number of segments before the mth one which are not shrunken.

Indeed, this pictorial representation of the action of $P^{(m,n)}$ generalises to the action on all boundary conditions if one represents $Q^{(M,N)}$ for arbitrary N by the union[8] of the sectors

[8]The decomposition of such a union into its constituents indeed corresponds to the D-brane charge

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$S_M \cup S_{M+1} \cup \ldots \cup S_{M+N-1}$ (c.f. figure 2b). This can be seen as follows. Consider first the situation, in which the pictorial operation deletes all the sectors belonging to the pictorial representation of a given boundary condition $Q^{(M,N)}$, i.e. the set $\{M, M+1, \ldots, M+N-1\}$ is completely contained in $\mathcal{L}^c_{(m,n)}$. In this case, by definition, $i(a) \geq N$ for all a, and hence by (5.30) $Q^{(M,N)}$ is annihilated by P. Thus, the pictorial action (5.39) agrees with the action of P. If on the other hand, the pictorial operation does not delete all segments belonging to the boundary condition $Q^{(M,N)}$, then as in (5.35)

$$\mathcal{L}_{(m,n)} \cap \{M, M+1, \ldots, M+N-1\} = \mathcal{J} \cap \{0, \ldots, N-1\} = \{i_1, \ldots, i_l\} \neq \emptyset \,, \tag{5.40}$$

and P does not annihilate $Q^{(M,N)}$. The result of the fusion has already been stated in (5.36). Since $n_i \geq 1$ for all i, we obviously obtain $k(a_1) = l$. But this is exactly the number of those segments of the pictorial representation of $Q^{(M,N)}$, which are not annihilated by P. Furthermore, a_1 is the number of $Q^{(M',1)}$ with $M' \in \{m, \ldots, M\}$ which are not annihilated by P. Thus, also in this case (5.39) applied to the pictorial representation of $Q^{(M,N)}$ is nothing but the pictorial representation of the result (5.36) of the fusion of P and $Q^{(M.N)}$. This shows that indeed $\mathcal{O}^{(m,n)}$ represents the action of $P^{(m,n)}$ on all boundary conditions.

In fact, a similar picture also describes the action of $P = P^{(m,n)}$ where $n_a = 0$ is allowed. In this case one has to replace the pictorial action (5.39) by

$$\widetilde{\mathcal{O}}^{(m,n)} = \widetilde{\mathcal{S}}_{(m,n)} \mathcal{T}_{-m} \,, \tag{5.41}$$

where now $\widetilde{\mathcal{S}}_{(m,n)}$ not only deletes all the segments S_M for which M is not in the image of the map

$$\widetilde{\imath}(a) = \sum_{j=1}^{a} n_j \,, \tag{5.42}$$

but in addition it also splits up every segment S_M into $|\widetilde{\imath}^{-1}(M)|$ segments. Thus, not only are $n_i - 1$ segments deleted for each i with $n_i > 1$, but also a new segment is created for each i with $n_i = 0$. For the flows between minimal model orbifolds however, only those defects $P^{(m,n)}$ with $n_i \geq 1$ play a role.

6. Defects and bulk flows between minimal model orbifolds

We propose that the defects presented in section 5 above arise in the way described in section 2 in supersymmetric bulk flows between orbifolds $\mathcal{M}_{d-2}/\mathbb{Z}_d$ of $N = 2$ superconformal minimal models. To give evidence for this proposal, we will analyse these flows in the mirror Landau-Ginzburg models in the following, and compare them to the fusion of the defects $P^{(m,n)}$ calculated in section 5.

6.1 Flows in the mirror Landau-Ginzburg models

As mentioned in section 3, in the mirror LG models, the flows we are interested in correspond to lower order deformations W_λ of the superpotential $W = W_{\lambda=0} = X^d$. We would

decomposition.

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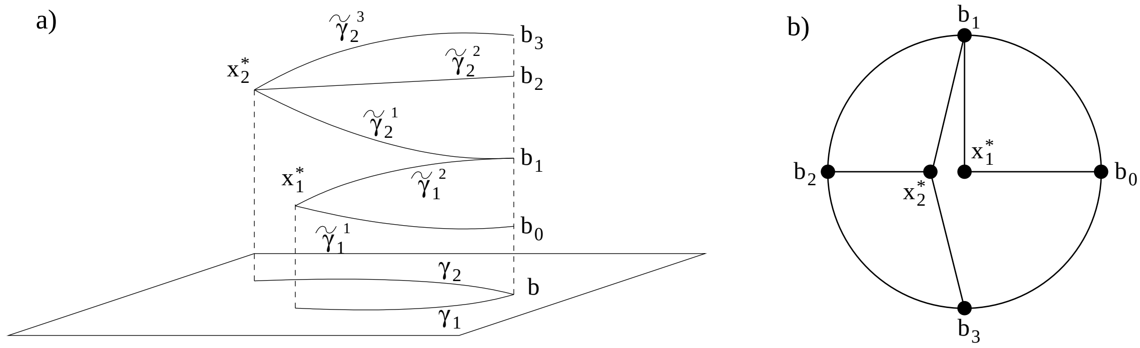

Figure 4: Polynomial of degree 4 with two critical points x_1^* and x_2^* of degrees $o(x_1^*) = 1$ and $o(x_2^*) = 2$ respectively. a) Paths γ_i between critical values $p(x_i^*)$ and base point b and their lifts $\tilde{\gamma}_i^\mu$ to the preimage of p. b) Schematic representation.

like to describe what happens to the corresponding A-type D-branes under such deformations. The relevant information about this is encoded in the structure of the critical points of the superpotential.

Let p be any polynomial of degree d in one variable. Regarded as a map $\mathbb{C} \to \mathbb{C}$, it is a d-sheeted branched cover of the complex plane. The branch points are the critical points x_i^* of p, in which $o(x_i^*) + 1$ many sheets meet. Here $o(x_i^*)$ denotes the order of the critical point. Let us choose a base point b near ∞ in the image of p, which is not a critical value. The preimage $p^{-1}(b)$ consists of d points which we denote by b_a, $a \in \mathbb{Z}_d$ in such a way that the monodromy around ∞ acts on the fiber over b by $b_a \mapsto b_{a+1}$.

Now let us suppose that all the critical values of p are different, and choose paths γ_i from the critical values $p(x_i^*)$ to b which only intersect each other in b. Then the preimage $p^{-1}(\gamma_i)$ consists of $o(x_i^*) + 1$ paths $\tilde{\gamma}_i^\mu$ going from x_i^* to $o(x_i^*) + 1$ distinct preimages $b_{a_i^\mu}$ of b (c.f. figure 4a). Taking b to ∞ and compactifying $\mathbb{C}$ to the disk, we obtain the following schematic representation (c.f. figure 4b). The points b_a are distinct points on the boundary of the disk, which are cyclically ordered, and each of the critical points x_i in the interior of the disk is connected by the $\tilde{\gamma}_i^\mu$ to $o(x_i^*) + 1$ of them. We call the union of these paths Γ_i. Γ_i and Γ_j for $i \neq j$ can only intersect on the boundary of the disk. Since $\sum_i o(x_i^*) = d - 1$ and all b_a have to be connected to each other on $\Gamma = \bigcup \Gamma_i$, each b_a can only lie on at most two different Γ_i, and Γ has to be simply connected, i.e. there are no closed loops on it.

Note however that this graphical representation depends on a choice of the (homotopy class of the) paths γ_i. In the following we will make a choice which is adapted to the description of A-branes in Landau-Ginzburg models. The latter are one-dimensional submanifolds of $\mathbb{C}$ on which the imaginary part $\Im(W)$ is constant and on which the real part $\Re(W)$ is bounded from below [27]. This means in particular that the world volumes of A-branes are unions $(-\tilde{\gamma}_i^\mu) \cup \tilde{\gamma}_i^{\mu'}$ of preimages under W of paths $\gamma_i = W(x_i^*) + \mathbb{R}^{\geq 0}$, where now x_i^* are the critical points of W. (A minus sign in front of a path indicates the inversion of the parametrisation or orientation.) Thus, if we assume that $\Im(W(x_i^*)) \neq \Im(W(x_j^*))$ for all $i \neq j$, this choice of paths γ_i gives rise to a schematic representation of A-branes in

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the LG model.

For instance for $W = X^d$, there is one critical point $x^* = 0$ of order $d-1$. The critical value $W(x^*) = 0$, thus A-branes consist of unions of two different premiages under W of the nonnegative real line $\mathbb{R}^{\geq 0}$, which are just $\widetilde{\gamma}^\mu = e^{\frac{2\pi i \mu}{d}}\mathbb{R}^{\geq 0}$ for $\mu \in \{0, \ldots, d-1\}$. The graphical representation is hence a disk with one point in the interior from which d lines representing $\widetilde{\gamma}^\mu$ go to the points b_μ on the boundary, and A-branes are unions $(-\widetilde{\gamma}^\mu) \cup \widetilde{\gamma}^\nu$ which we will denote by $\overline{b_\mu x^* b_\nu}$.

Under a deformation W_λ of W, the critical point x^* splits up into N distinct critical points x_i^*. If we assume that for all $\lambda > 0$ the imaginary parts $\Im(W(x_i^*))$ are all distinct, and no further splitting of critical points occurs, then the "topology" of the graphical representation does not change.

The renormalisation group flow now drives W_λ to a homogeneous superpotential, i.e. at its endpoint, there is only a single critical point left at 0. The other critical points go off to ∞. If under the RG flow the imaginary parts $\Im(W(x_i^*))$ of the critical values all stay separate and no further splitting of critical points occur, than it is easy to see what happens to A-branes under this perturbation. A-branes which are attached to critical points x_i^*, $i > 1$ going off to ∞ decouple[9] from the theory, while A-branes attached to the critical point x_1^* which remains finite flow to the respective A-branes in the IR. A-branes consisting of rays which are separated by the perturbation, i.e. rays which emanate from different critical points for $\lambda \neq 0$ have to decay into sums of A-branes of the two types above by addition and subtraction[10] of rays going to those boundary points b_a which lie on intersections of graphs Γ_i and Γ_j. The summands then behave as described above. For the special class of perturbations $W_\lambda = X^{nd} + \lambda X^d$ this has been analysed in detail in [35].

This flow on A-branes has a simple description in terms of the graphical representation of the deformations W_λ. In the UV, A-branes $\overline{b_i x^* b_j}$ are specified by pairs (b_i, b_j) of two different boundary points. The same is true in the IR, where however only the boundary points $b_{a_1^\mu}$ remain. The flow associated to a graphical representation on the level of A-branes is then just described by identifying all boundary points $b_i \sim b_j$ which are connected on $\Gamma - \Gamma_1$. An A-brane (b_i, b_j) in the UV therefore flows to the brane $([b_i], [b_j])$ in the IR, where $[\cdot]$ denotes the equivalence class with respect to the equivalence relation $\sim$. If in particular the two points (b_i, b_j) defining an A-brane in the UV are identified by $\sim$ then the brane decouples from the theory. Note that while the set $\{[b_i]\}$ of rest classes forms a cyclically ordered set, there is an ambiguity of identifying it with $\mathbb{Z}_{d'}$. The latter is related to the freedom of a quantum symmetry operation in the IR.

As a simple example let us consider the perturbation corresponding to $W_\lambda = X^d + \lambda X^{d-1}$. The corresponding RG flow drives the system from the LG model with superpotential $W = X^d$ to the one with $W = X^{d-1}$ (c.f. section 3). The critical points of W_λ are $x_1^* = 0$ of order $d-2$ and $x_2^* = -\lambda$ of order 1. Thus, for $\lambda \neq 0$ a critical point x_2^*

[9]Their bulk-boundary couplings go to zero in the IR. This is clear for the topological couplings. Since the flows at hand preserve supersymmetry also in the boundary sectors, and since furthermore the supersymmetric boundary conditions in these models are classified, it also follows for all bulk-boundary couplings on the level of the full conformal field theory.

[10]Addition with opposite orientation.

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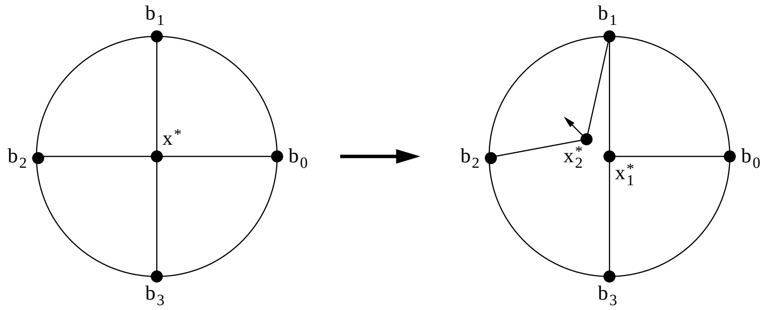

Figure 5: Flow corresponding to $W_\lambda = X^4 + \lambda X^3$ for a particular choice of λ.

of order 1 splits off from the critical point in 0 and goes to ∞ under the RG flow. The A-brane $\overline{b_{a_2^1} x_2^* b_{a_2^2}}$ consisting of the two preimages $\widetilde{\gamma}_2^1$ and $\widetilde{\gamma}_2^2$ of γ_2 decouple from the theory, while A-branes consisting of preimages $\widetilde{\gamma}_1^\mu$ of γ_1 flow to the corresponding A-branes in the IR. All other A-branes decay into sums of A-branes of the two types. More precisely, if Γ_1 and Γ_2 intersect in $b_{a_2^1} = b_{a_1^\nu}$, then A-branes $\overline{b_{a_2^2} x^* b_{a_1^\mu}}$ in the UV decay into sums $\overline{b_{a_2^2} x^* b_{a_2^1}} + \overline{b_{a_1^\nu} x^* b_{a_1^\mu}}$ whose first summand decouples in the IR, while the second one stays in the theory. For the case $d = 4$ this is schematically represented in figure 5. Which of the rays $\overline{x^* b_a}$ is torn off the UV critical point, and whether b_a is connected to b_{a-1} or b_{a+1} by Γ_2 depends on the phase of the perturbation parameter λ.

More generally, the topology of the graphical representation of a deformation depends on the form of W_λ in a complicated way. Since the graphical representation carries the information relevant for the analysis of the behaviour of A-branes under the respective flows, we will avoid working directly with the deformations W_λ of the superpotential in the following, but instead characterise a perturbation directly by the graphical representation.

6.2 Comparison

Indeed, the graphical representation of the behaviour of A-branes under bulk flows in the mirror LG models described in the previous section is very reminiscent of the operation of the defects $P^{(m,n)}$ on B-branes in the corresponding LG orbifolds. In section 5.4 above, we gave a pictorial representation of B-branes in the LG-orbifolds $X^d/\mathbb{Z}_d$, in which the B-brane associated to a matrix factorisation $Q^{(M,N)}$ was represented by a union of consecutive segments $S_M \cup \ldots \cup S_{M+N-1}$ of a disk divided into d segments. It can be easily worked out that the graphical representation of the corresponding mirror A-brane in the unorbifolded LG model with superpotential $W = X^d$ is given by $\overline{b_M x^* b_{M+N}}$. Thus, the mirror map just replaces a union of consecutive segments by its oriented boundary.

It is now obvious that under the mirror map bulk flows of A-branes encoded in graphical representation $\{\Gamma_i\}$ such as in figure 6 can be pictorially represented by shrinking sectors, and can therefore be described by defects $P^{(m,n)}$. More precisely, let

$$\mathcal{L} = \{a \in \mathbb{Z}_d \,|\, b_a \nsim b_{a+1}\}\,, \tag{6.1}$$

be the set of neighbouring points b_a which are not connected on $\Gamma - \Gamma_1$, and denote the complement by $\mathcal{L}^c$. Then the corresponding flow on the B-side can be represented

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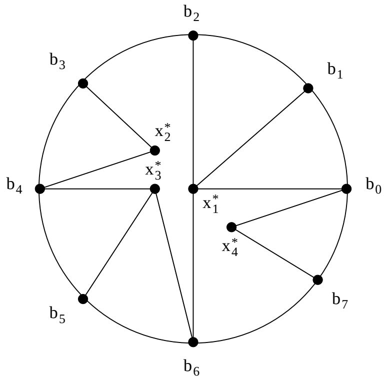

Figure 6: Representation of a deformation of a degree 8 polynomial with four critical points.

pictorially by the shrinking operation $\mathcal{S}_{\mathcal{L}^c}$ defined in section 5.4, and thus it is realised by the corresponding defect $P^{(m,n)}$ with

$$\mathcal{L} = m + \{0, n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{o(x_1^*)+1}\}\,. \tag{6.2}$$

Note that this parametrisation of $\mathcal{L}$ is ambiguous. Namely one can shift $m \mapsto m + \sum_{i=1}^{j} n_i$ and change the n_i accordingly. This operation is nothing else than the IR quantum symmetry, which we identified above as giving rise to an ambiguity of the flow on A-branes.

For instance, the perturbation corresponding to figure 6 can be represented on the B-side by the shrinking operation $\mathcal{S}_{\{3,4,5,7\}}$ and is therefore described by the defect $P^{(0,(1,1,4,2))}$.

To summarise, the analysis of the induced flows of A-branes the mirror Landau-Ginzburg models indeed confirms that the defects $P^{(m,n)}$ describe the flows between minimal model orbifolds.

7. Flows between $\mathbb{C}/\mathbb{Z}_d$-orbifolds

There is a close link between Landau-Ginzburg models with superpotential W and non-compact affine orbifold theories that can be obtained from the former by letting the superpotential go to zero. Although the theories are in fact quite different, for example have different central charge and F-terms, the structure of their twisted chiral sectors (twisted F-terms) is unaffected by the presence or absence of an untwisted chiral superpotential [36].

Thus, the discussion of twisted chiral perturbations of Landau-Ginzburg orbifolds above carries over to the case of affine orbifold models of type $\mathbb{C}/\mathbb{Z}_d$, which can be regarded as $\mathbb{Z}_d$ orbifolds of Landau-Ginzburg models of a single chiral superfield with superpotential $W = 0$. Indeed, these models have the same twisted chiral rings as the minimal model orbifolds $\mathcal{M}_{d-2}/\mathbb{Z}_d$ with one (a, c) field coming from the ground state of each twisted sector. The perturbations we have been studying for the minimal model are hence directly related to the perturbations by twisted chiral fields in the affine orbifold models. Indeed, it has been found in [2, 1] that non-supersymmetric orbifold singularities of type $\mathbb{C}/\mathbb{Z}_d$ flow under perturbation by twisted chiral fields into a number of disconnected lower orbifold singularities in the IR, which is analogous to what one finds for the minimal model orbifolds. In

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fact, there is a common treatment of the corresponding flows $\mathcal{M}_{d-2}/\mathbb{Z}_d \mapsto \mathcal{M}_{d'-2}/\mathbb{Z}_{d'-2}$ between minimal model orbifolds and $\mathbb{C}/\mathbb{Z}_d \mapsto \mathbb{C}/\mathbb{Z}_{d'}$ between affine orbifolds in the framework of gauged linear sigma models (see e.g. [3]). This suggests that also the flows between affine orbifolds can be described by defects with a structure similar to that of the $P^{(m,n)}$.

Generally, matrix factorisations P of any polynomial W give rise to matrix factorisations of $W = 0$ by setting $p_0 = 0$. In this way, from the matrix factorisations $P^{(m,n)}$ defining defects between LG orbifolds $X^d/\mathbb{Z}_d$ one also obtains defects between orbifolds $\mathbb{C}/\mathbb{Z}_d$. Obviously they obey the same fusion algebra as the ones in the LG orbifolds, and also their action on B-type defects is similar. Therefore these defects are the natural candidates to describe the corresponding flows between the affine orbifolds.

As a side remark we would like to mention that for Landau-Ginzburg (orbifold) models with superpotential $W = 0$, fusion with a defect corresponding to a matrix factorisation $P = (p_1, 0)$ can also be thought of as Fourier-Mukai transform with kernel the (equivariant) sheaf associated to the module $\mathrm{coker}(p_1)$. This description is more in line with the common description of D-branes in these models in terms of (equivariant) coherent sheaves.

8. Discussion

In this paper, we have considered the behaviour of B-type D-branes in $\mathbb{Z}_d$-orbifolds of $N = 2$ minimal models $\mathcal{M}_{d-2}$ under bulk perturbations generated by relevant twisted chiral operators. The new approach put forward here is based on the idea that perturbations of conformal field theories give rise to defect lines between the UV and the IR theory of the corresponding renormalisation group flows. This turns out to be particularly useful in the treatment of bulk perturbations on surfaces with boundaries. Namely, the effect of bulk flows on the boundary conditions can then be realised by merging this defect with the respective boundary condition of the UV theory to obtain a new boundary condition of the IR theory. A related idea has been put forward in [10], where it was shown how certain boundary RG flows can be universally induced by fusion with defects. In situations where $N = 2$ supersymmetry is preserved, the fusion procedure can be performed on the level of the respective topologically twisted theories, making it unnecessary to deal with regularisation issues.

Using the Landau-Ginzburg representation, we constructed a set of B-type defects between minimal model orbifolds as equivariant matrix factorisations of the difference of the respective superpotentials, and we proposed them to be associated to bulk flows between these models. Their fusion among themselves and with B-type boundary conditions was easily computed in the matrix factorisation formalism, and a comparison with the chiral perturbations of the mirror LG models confirms that the defects indeed have the correct properties to describe the flows.

We also argued that in an analogous way one can construct similar defects which describe corresponding flows between affine orbifolds of type $\mathbb{C}/\mathbb{Z}_d$.

Having obtained defects arising in flows between Landau-Ginzburg (or affine) orbifolds with a single chiral superfield, it would be very interesting to find defects describing flows between such models with several variables. In these models the analysis of the behaviour

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of A-branes under the corresponding flow in the mirror theories is much more complicated, so that the defect approach would be very useful. It would allow the explicit computation of flows of B-branes under bulk perturbations for instance in the $\mathbb{C}^2/\Gamma$-orbifolds studied in [2, 4 – 6].

Besides these special examples, we expect our approach to be powerful in any situations where world sheet supersymmetry (as opposed to space-time supersymmetry) is preserved. The extension to non-supersymmetric theories, or an understanding of our flow defects on the level of the full conformal field theory as opposed to its topological subsector, is less straight forward, because it requires a regularisation procedure for the fusion of non-topological defects with boundary conditions. The investigation of the fusion properties of non-topological defects on the level of the full conformal field theory has recently been started in [25] for the example of the free boson. One of the conclusions of that paper was that non-topological defects are generically unstable and tend to decay via defect-dissociation, the inverse process of fusion. The defects investigated in the current paper are certainly non-topological on the level of the full conformal field theory, and one might wonder what possible decay channels could arise. A part of the answer is already given in section 5, where we have shown that defects between the minimal models $\mathcal{M}_{d+n-2}$ and $\mathcal{M}_{d-2}$ can be obtained by fusing n single step defects that relate $\mathcal{M}_{d+i-1}$ and $\mathcal{M}_{d+i-2}$. It would be an interesting problem to determine via an analysis of the defect entropy proposed in [25] wether our defects tend to dissociate into smaller step operators.

While we focused on relevant perturbations in this paper, by the same reasoning defects can also be used in the study of exactly marginal bulk perturbations. (These do not necessarily stay marginal in the presence of boundaries [21] but can induce non-trivial RG flows in the boundary sectors.) For instance, σ-models on Calabi-Yau target spaces have no tachyons, and hence do not exhibit relevant perturbations. But they do allow for exactly marginal perturbations in general. Deforming such a theory around a singularity in its Kähler moduli space, the corresponding monodromy transformation on the B-type D-branes should be described by a defect. Indeed this is not at all surprising, because these transformation can be represented as Fourier-Mukai transformations (see e.g. [37]), which at least on a superficial level are related to defects via the folding trick. The defect representation of these transformations in the Landau-Ginzburg phase is formulated in [38], see also [39].

Even though there are no relevant flows between different Calabi-Yau σ-models, it is still possible to construct defects between such models, for instance using the ones constructed in this paper for single minimal model orbifolds as building blocks. The physical meaning of such defects however is unclear. Clearly they relate different string vacua, and one might speculate that defect transitions could require some meaning, e.g. as tunneling amplitudes in a background independent formulation of string theory.

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A. The conformal field theory point of view

In this appendix we would like to discuss various features that appeared in the main text from the point of view of the conformal field theory, to which the LG model flows in the IR. Our discussion will be restricted to defects that preserve the superconformal symmetry, in particular, we will only consider defects between one minimal model and itself.

In the IR, the Landau-Ginzburg model with one chiral superfield and superpotential $W(X) = X^d$ flows to the unitary superconformal minimal model $\mathcal{M}_k$, $k = d-2$ with A-type modular invariant partition function. These conformal field theories are rational with respect to the $N = 2$ super Virasoro algebra at central charge $c_k = \frac{3k}{k+2}$. In fact, the bosonic part of this algebra can be realised as the coset W-algebra

$$(\mathrm{SVir}_{c_k})_{\mathrm{bos}} = \frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{u}}(1)_4}{\widehat{\mathfrak{u}}(1)_{2k+4}}\,, \tag{A.1}$$

and the respective coset CFT can be obtained from $\mathcal{M}_k$ by a non-chiral GSO projection.

The Hilbert space $\mathcal{H}^k$ of $\mathcal{M}_k$ decomposes into irreducible highest weight representations of holomorphic and antiholomorphic super Virasoro algebras, but it is convenient to decompose it further into irreducible highest weight representations $\mathcal{V}_{[l,m,s]}$ of the bosonic subalgebra (A.1). These representations are labelled by

$$[l,m,s] \in \mathcal{I}_k := \{(l,m,s)\,|\,0 \le l \le k,\ m \in \mathbb{Z}_{2k+4},\ s \in \mathbb{Z}_4,\ l+m+s \in 2\mathbb{Z}\}/\sim\,, \tag{A.2}$$

where $[l,m,s] \sim [k-l, m+k+2, s+2]$ is the field identification. The highest weight representations of the full super Virasoro algebra are given by

$$\mathcal{V}_{[l,m]} := \mathcal{V}_{[l,m,(l+m)\,\mathrm{mod}\,2]} \oplus \mathcal{V}_{[l,m,(l+m)\,\mathrm{mod}\,2+2]}\,. \tag{A.3}$$

For $(l+m)$ even $\mathcal{V}_{[l,m]}$ is in the NS-, for $(l+m)$ odd in the R-sector. Here $[l,m] \in \mathcal{J}_k := \{(l,m)\,|\,0 \le l \le k,\ m \in \mathbb{Z}_{2k+4}\}/\sim$, $[l,m] \sim [k-l, m+k+2]$. The Hilbert spaces of $\mathcal{M}_k$ in the NSNS- and RR-sectors then read

$$\mathcal{H}^k_{\mathrm{NSNS}} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\mathrm{even}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,m]}\,, \qquad \mathcal{H}^k_{\mathrm{RR}} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\mathrm{odd}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,m]}\,. \tag{A.4}$$

The theory exhibits an action of a $\mathbb{Z}_{k+2}$ symmetry group, realised by the simple current $(0,2,0)$. Orbifolding by this group introduces twisted sectors, in which the representations of the left- and right-movers differ by the action of the appropriate power of the simple current. Having included the twisted sectors, one has to projects onto $\mathbb{Z}_{k+2}$-invariant sectors to obtain the Hilbert space of the orbifold theory. The action of the generator of

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the orbifold group in the twisted sector $\psi \in \mathcal{V}_{[l,m,s]} \otimes \mathcal{V}_{[l,m-2n,s]}$ is given by multiplication with the phase

$$\psi \mapsto e^{2\pi i \frac{m+m-2n}{2(k+2)}} \psi \,. \tag{A.5}$$

The resulting Hilbert space differs from the initial unorbifolded one only by a relative minus sign of m-labels in the left- and right-moving sectors:

$$\mathcal{H}^k_{\mathrm{NSNS}} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k\\ l+m\,\mathrm{even}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,-m]}\,, \qquad \mathcal{H}^k_{\mathrm{RR}} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k\\ l+m\,\mathrm{odd}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,-m]}\,. \tag{A.6}$$

A.1 Defects

B-type defects in minimal models have been considered in [18]. Here, defects were formulated as maps between closed string Hilbert spaces. They can be written as sums over projectors onto modules of the bosonic subalgebra of the full supersymmetric model:

$$\mathcal{D} = \sum_{\substack{[l,m,s],\bar{s}\\ s-\bar{s}\,\mathrm{even}}} \mathcal{D}^{[l,m,s,\bar{s}]} \mathrm{P}_{[l,m,s,\bar{s}]}\,, \tag{A.7}$$

where $\mathrm{P}_{[l,m,s,\bar{s}]}$ is a projector on the subspace $\mathcal{V}_{[l,m,s]} \otimes \mathcal{V}_{[l,m,\bar{s}]}$ of the Hilbert space. It is furthermore understood that

$$\mathcal{D}^{[l,m,s+2,\bar{s}]} = \eta \mathcal{D}^{[l,m,s,\bar{s}]} \quad \text{and} \quad \mathcal{D}^{[l,m,s,\bar{s}+2]} = \bar{\eta} \mathcal{D}^{[l,m,s,\bar{s}]}\,. \tag{A.8}$$

Consistent choices for the prefactors of the projection operators are given by

$$\mathcal{D}^{[l,m,s,\bar{s}]}_{[L,M,S,\bar{S}]} = e^{-i\pi\frac{\bar{S}(s+\bar{s})}{2}} \frac{S_{[L,M,S-\bar{S}][l,m,s]}}{S_{[0,0,0],[l,m,s]}}\,, \tag{A.9}$$

where the different defects have been labelled by $[L,M,S,\bar{S}]$ with $[L,M,S-\bar{S}] \in \mathcal{I}_k$, and

$$S_{[L,M,S][l,m,s]} = \frac{1}{k+2} e^{-i\pi\frac{Ss}{2}} e^{i\pi\frac{Mm}{k+2}} \sin\left(\pi \frac{(L+1)(l+1)}{k+2}\right) \tag{A.10}$$

is the modular S-matrix for the coset representations $\mathcal{V}_{[l,m,s]}$. It is then straightforward to determine the composition of defects and their action on boundary states [18].

To obtain the defect in the orbifold theory, one simply has to switch the sign of m for the right movers, such that the defect reads

$$\mathcal{D}^{\mathrm{orb}} = \sum_{\substack{[l,m,s],\bar{s}\\ s-\bar{s}\,\mathrm{even}}} \mathcal{D}^{[l,m,s,\bar{s}]} \mathrm{P}^{-}_{[l,m,s,\bar{s}]}\,, \tag{A.11}$$

where $\mathrm{P}^{-}_{[l,m,s,\bar{s}]}$ is a projector on the subspace $\mathcal{V}_{[l,m,s]} \otimes \mathcal{V}_{[l,-m,\bar{s}]}$ of the Hilbert space.

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A.2 The folding trick

The folding trick relates defects to permutation boundary states [40] of the tensor product of two minimal models. We start with the unorbifolded theory. B-type permutation boundary states in a tensor product of two minimal models satisfy the following conditions

$$\begin{aligned} \left(G_r^{\pm(1)} + i\eta_1 \bar{G}_{-r}^{\pm(2)}\right)\|B\rangle\rangle &= 0 \\ \left(G_r^{\pm(2)} + i\eta_2 \bar{G}_{-r}^{\pm(1)}\right)\|B\rangle\rangle &= 0 \end{aligned} \tag{A.12}$$

In the case $\eta_1 = \eta_2$ the boundary conditions preserve the diagonal $N = 2$ algebra. The corresponding boundary states have been discussed in [41, 42] and are explicitely given by

$$\|[L, M, S_1, S_2]\rangle\rangle = \frac{1}{2\sqrt{2}} \sum_{l,m,s_1,s_2} \frac{S_{Ll}}{S_{0l}} e^{\pi i M m/(k+2)} e^{-i\pi(S_1 s_1 - S_2 s_2)/2} |[l, m, s_1] \otimes [l, -m, -s_2]\rangle\rangle \tag{A.13}$$

Permutation boundary states in minimal model orbifolds can now be constructed using standard conformal field theory techniques. We first note that the B-type permutation boundary states (A.13) in the unorbifolded theory are invariant under the diagonal subgroup $\mathbb{Z}_d \subset \mathbb{Z}_d \times \mathbb{Z}_d$ generated by the product $g = g_1 g_2$ of the generators of the two $\mathbb{Z}_d$'s. To construct the g^n-twisted components of the boundary states we observe that the permutation gluing condition requires that $\bar{m}_2 = -m_1$ and $m_2 = -\bar{m}_1$. In the sector twisted by g^n the relation between left- and right-moving m-labels is $m_1 = \bar{m}_1 + 2n$, $m_2 = \bar{m}_2 + 2n$, so that the relevant Ishibashi states have labels $m_2 = -\bar{m}_1 = -m_1 + 2n$. Therefore, the twisted boundary states take the form

$$\begin{aligned} \|L, M, \hat{M}, S_1, S_2\rangle\rangle_{(-1)^{(s+1)F} g^n} = \; & \frac{1}{2} e^{-\frac{\pi i n}{k+2}(M+\hat{M})} \sum_{l,m} \sum_{\nu,\nu_2 \in \mathbb{Z}_2} \frac{S_{Ll}}{S_{0l}} e^{\pi i \frac{Mm}{k+2}} (-1)^{S_1\nu_1 + S_2\nu_2} \\ & e^{-\pi i \frac{s}{2}(S_1+S_2)} |[l, m, s + 2\nu_1] \otimes [l, -m + 2n, s + 2\nu_2]\rangle\rangle \,, \end{aligned}$$

where the additional label $\hat{M}$ specifies the representation of the diagonal $\mathbb{Z}_d$ on the Chan-Paton factors. The subscript denotes the twist: for g^n the Ishibashi states are in the n^{th} twisted sector. Furthermore, s distinguishes between NS and R sector, in our notation the NS sector is the $(-1)^F$ twisted R-sector. We require that $M + \hat{M}$ is always even, so that the boundary state is invariant under $n \to n + k + 2$. Also, as before, to preserve the diagonal $N = 2$ we require that $L + M$ and $S_1 + S_2$ are even.

To obtain a boundary state that is invariant under the full $\mathbb{Z}_d \times \mathbb{Z}_d$ orbifold group, we need to perform the projection (A.5) on states with $2m = 2n \bmod 2k + 4$. This yields the following boundary state

$$\begin{aligned} \|L, \hat{M}, S_1, S_2\rangle\rangle_{(-1)^{(s+1)F}} = \; & \frac{1}{2} \sum_{l,m} \sum_{\nu,\nu_2 \in \mathbb{Z}_2} \frac{S_{Ll}}{S_{0l}} e^{\pi i \frac{\hat{M}m}{k+2}} (-1)^{S_1\nu_1 + S_2\nu_2} \\ & e^{-\pi i \frac{s}{2}(S_1+S_2)} |[l, m, s + 2\nu_1] \otimes [l, m, s + 2\nu_2]\rangle\rangle \,. \end{aligned}$$

In this way we have constructed B-type permutation boundary states in the tensor product of minimal model orbifolds out of those in the corresponding unorbifolded theory. This

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orbifold procedure is analogous to the one described on the level of Landau-Ginzburg models in section 4. In particular, after unfolding the states with $L = 0$ correspond to the defects realising the group of quantum symmetries in the orbifold theory which have been constructed in Landau Ginzburg formalism in section 4.3.

A.3 Cylinder amplitude and the folding trick

From the formula of the defect operators A.11, it is straighforward to determine the fusion of the corresponding defects with D-branes. Instead of doing this calculation, we find it instructive to present an alternative derivation using the folding trick. More specifically, we will compute cylinder amplitudes in the tensor product of minimal models between permutation boundary states on one side and tensor product boundary states on the other. Via the folding trick we will reinterprete them as cylinder amplitudes in a single minimal model with boundary conditions corresponding to the two tensor factors on both ends of the cylinder with a defect line corresponding to the permutation boundary state in between them. The relevant one-loop amplitude in the unorbifolded theory is

$$\begin{aligned}
&\langle\langle (L_1,S_1)\| \otimes \langle\langle (L_2,S_2)\| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} \; \| \, [\hat{L},\hat{M},\hat{S}_1,\hat{S}_2]\rangle\rangle \qquad\qquad (A.14)\\
&\quad = \sum_{[l,m,s]} \chi_{[l,m,s]}(\tilde{q}^{1/2}) \sum_{\hat{l}} \Big(N_{L_1 L_2}{}^{\hat{l}}\, N_{\hat{l}\hat{L}}{}^{l}\, \delta^{(4)}(s+\hat{S}_1+\hat{S}_2-(S_1+S_2)+1)\\
&\qquad\qquad + N_{k-L_1 L_2}{}^{\hat{l}}\, N_{\hat{l}\hat{L}}{}^{l}\, \delta^{(4)}(s+\hat{S}_1+\hat{S}_2-(S_1+S_2)-1)\Big)\,.
\end{aligned}$$

Here $\|(L_i,S_i)\rangle\rangle$ are B-type boundary state in a single minimal model (see e.g. [18] for more details on the notation). Note that in the case $\hat{S}_1 = \hat{S}_2 \bmod 2$ and $S_1 = S_2 \bmod 2$ the representations appearing in the open string sector are formally in the R-sector, but are to be interpreted as twisted NS-sector representations. In the closed string sector, this shift is related to the fact that the overlap of a tensor product with a permutation boundary state is a trace with an insertion of the permutation σ. Since σ interchanges states, a minus sign is picked up in the fermionic relative to the bosonic case.

$$\begin{aligned}
\langle\langle [l,0,s]\otimes[l,0,s]| q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} |[l,0,s]\otimes[l,0,s]\rangle\rangle^{\sigma} &= \mathrm{Tr}_{[l,0,s]\otimes[l,0,s]}\left(q^{L_0-\frac{c}{12}}\,\sigma\right)\\
&= e^{-\pi i s/2}\chi_{[l,0,s]}(q^2)\,. \qquad (A.15)
\end{aligned}$$

We would now like to find the defect $\mathcal{D}$ corresponding to the permutation boundary state. This can be deduced by comparing the characters appearing in this cylinder amplitude with those appearing in the cylinder amplitudes between two B-type boundary states in a single minimal model. The goal is to find a homomorphism $\mathcal{D}$ such that the above amplitude is reproduced by the cylinder amplitude between $\mathcal{D}\|(L_1,S_1)\rangle\rangle$ on one side and $\|(L_2,-S_2)\rangle\rangle$ on the other. Note that in the cylinder amplitude taken in the tensor product, the boundary states $\|(L_1,S_1)\rangle\rangle$ and $\|(L_2,S_2)\rangle\rangle$ are both ingoing (or both outgoing). On the other hand, taking a cylinder amplitude in a single minimal model, one of the boundaries becomes outgoing (ingoing), so that one of the states has to be conjugated: $\|(L,S)\rangle\rangle \mapsto \langle\langle(L,-S)\|$. Further care must be taken because of the phase $(-1)^{F_L}$ that appears in the folded model. Taking this phase into account effectively shifts the S-label of a B-type boundary state by

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one (changing the spin structure), such that the $\delta^{(4)}$-constraint in the above formula gets shifted by one. Taking all of this into account, the above formula is consistent with the defect action

$$\begin{aligned}\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}\|[L_2,M_2,S_2]\rangle\!\rangle_B &= \sum_{[L,M,S]\in\mathcal{I}_k} \mathcal{N}_{[L_1,M_1,S_1-\bar{S}_1][L_2,M_2,S_2]}^{[L,M,S]}\|[L,M,S]\rangle\!\rangle_B \\ &= \sum_L \mathcal{N}_{L_1L_2}^{L}\|[L,M_1+M_2,S_1-\bar{S}_1+S_2]\rangle\!\rangle_B\,,\end{aligned} \tag{A.16}$$

such that the permutation boundary state corresponds to the defect operator (A.7) with the same labels. The discussion in the orbifold theory is similar, the only difference being that the boundary states of the orbifold have an additional M-label, leading to a $\delta^{(2k+4)}$ constraint on the m-labels in all cylinder amplitudes.

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